Optimal Regulation of Virtual Power Plants

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Abstract—This paper develops a real-time algorithmic framework for aggregations of distributed energy resources (DERs) in distribution networks to provide regulation services in response to transmission-level requests. Leveraging online primal-dual-type methods for time-varying optimization problems and suitable linearizations of the nonlinear AC power-flow equations, we believe this work establishes a system-theoretic foundation to realize the vision of distribution-level virtual power plants. The optimization framework controls the output powers of dispatchable DERs such that, in aggregate, they respond to automatic generation control and/or regulation-services commands. This is achieved while concurrently regulating voltages within the feeder and maximizing customers’ and utility’s performance objectives. Convergence and tracking capabilities are analytically established under suitable modeling assumptions. Simulations are provided to validate the proposed approach.

Distribution systems, virtual power plants, real-time optimization, optimization with feedback.

I. INTRODUCTION

Traditional approaches for regulating frequency and maintaining reliable operation of transmission systems leverage primary frequency response, automatic generation control (AGC), and regulation services provided by large-scale synchronous generators. In the future, dispatchable distributed energy resources (DERs) are envisioned to supplement generation-side capabilities, by providing additional flexibility in regulating frequency and maintaining reliable system operation [1], [2]. Towards realizing this vision, we develop an algorithmic framework for DER aggregations in distribution feeders to emulate a virtual power plant that effectively provides regulation services to the bulk system while guaranteeing power quality in the distribution network.

The main idea and technical approach are outlined with respect to the illustrative system in Fig. 1. The objective is to develop a real-time optimization architecture for DERs, so that the active power at the feeder head, $P_0$, is adjusted in real-time to track setpoint $P_{0,\text{set}}$ (while we focus on active power, the framework can be extended to consider reactive-power setpoints too). For example, $P_{0,\text{set}}$ can be an AGC signal (scaled by a given feeder participation factor), a ramping signal, or a 5-minute dispatch commanded by the transmission system operator (e.g., flexible ramping products) [2]. The real-time algorithm is designed to track the setpoint $P_{0,\text{set}}$ at the feeder head, while concurrently: i) maximizing customers’ and utility’s performance objectives, and ii) ensuring that operational limits are enforced throughout the feeder. The algorithm is developed with the aid of online primal-dual-gradient methods applied to double-smoothed Lagrangian functions [3], [4] while relying on suitable linear approximations of the AC power-flow equations to bypass their nonlinearity. As shown in Fig. 1, the resultant operational strategy involves collecting measurements of pertinent voltages and powers in real-time. These measurements and the setpoint $P_{0,\text{set}}$ are then utilized to dispatch individual DERs. Convergence and tracking capabilities of the proposed algorithms are analytically established.

Prior works in this context have considered controlling aggregations of DERs such as thermostatically controlled loads [5]–[7] and other deferrable loads (e.g., pool pumps) [8] to track given power setpoints at the substation. A control framework for scheduling and provisioning of frequency reserves by aggregations of commercial buildings is proposed in [9]. Strategies to manage fleets of electric vehicles to provide services to the main grid are investigated in [10]. A framework for modeling of flexible loads as virtual batteries...
is proposed in [11]. However, the optimization and control strategies proposed in [5]–[10] are network-agnostic, in the sense that they assume that all DERs are connected to one electrical node and power flows in the network are ignored. Further, analytical tracking results in time-varying operational conditions are not available in the majority of those works. While this simplifies the design of optimization and control strategies, such strategies do not allow for the regulation of voltage levels throughout the feeder. On the other hand, the proposed methodology enables tracking of power setpoints at the substation while enforcing voltage limits. An economic dispatch model is considered in [12], where distributed controllers are designed to meet a certain load profile over a finite time horizon while minimizing an aggregate cost; stability of the distributed algorithm is analyzed in a continuous-time setting (and, hence, the effects of communication delays and discrete-time operations are not included) and for time-invariant conditions.

A virtual-power-plant profit-maximization problem over a 24-hour period is considered in [13], based on an economic-dispatch model, and a stochastic adaptive optimization approach for virtual power plants participating in the day-ahead and the real-time energy markets is proposed in [14]. However, [13], [14] require solving optimization problems to convergence (i.e., they provide an offline solution method), and are not applicable to real-time control of DERs. A centralized real-time controller for distribution systems is proposed in [15], [16], based on projected-gradient methods; it is shown that the closed-loop system converges on average to a predetermined objective. Compared to [15], [16], our proposed framework affords a distributed implementation, it is based on an online primal-dual method (instead of barrier-type functions, which may lead to prolonged constraint violations), and it is shown to be stable under dynamic operational conditions. The online algorithm in [17] does not utilize measurements of voltage magnitudes and powers at the substation; therefore, it may nor enforce voltage regulation and it may fail in tracking setpoints at the substation. Furthermore, the algorithm in [17] considers a diminishing stepsize rule, which is not suitable for the dynamic setting considered in the present paper.

A similar controller design strategy is proposed in [18]; however, the algorithm in [18] does not utilize measurements of the output powers of the DERs. In contrast, the proposed method is based on measurements of the DERs’ output powers to promote adaptability and cope with slow-responding DERs. Compared to [18], the present paper provides suitable convergence claims for the considered tracking problem, it provides a more general approximate model for voltage magnitudes, and it provides an approximate model for the power flows at the substation. Finally, a real-time distributed algorithm for the optimal power flow problem is proposed in [19]; however, the distributed algorithm relies on nested loops (thus imposing stringent communication constraints) and it is applicable only to radial (balanced) distribution systems. Relative to [19], the proposed method requires a much simpler communication strategy, and it is applicable to generic systems with mesh or radial topologies.

The remainder of this manuscript is organized as follows. Section II outlines preliminaries and system model. The real-time algorithm is described in Section III; numerical results are presented in Section IV, and concluding remarks are in Section V. Additional modeling details, an extension to multiphase systems, and proofs are provided in the Appendix.

II. PRELIMINARIES

A. Distribution-network Model

Consider a distribution feeder comprising \(N + 1\) nodes collected in the set \(\mathcal{N} \cup \{0\}, \mathcal{N} := \{1, \ldots, N\}\), and lines represented by the set of edges \(\mathcal{E} := \{(m, n)\} \subset (\mathcal{N} \cup \{0\}) \times (\mathcal{N} \cup \{0\})\).\(^1\) Let \(V_n \in \mathbb{C}\) and \(I_n \in \mathbb{C}\) denote the phasors for the voltage and the current injected into node \(n\), respectively, and define the \(N\)-dimensional complex vectors \(\mathbf{v} := [V_1, \ldots, V_N]^T \in \mathbb{C}^N\) and \(i := [I_1, \ldots, I_N]^T \in \mathbb{C}^N\). Node 0 denotes the secondary of the distribution transformer. Using Ohm’s and Kirchhoff’s circuit laws, the following relationship can be established:

\[
\begin{bmatrix}
I_0
i
\end{bmatrix} = \begin{bmatrix}
y_{00}
y^T
\end{bmatrix} \begin{bmatrix}
Y
\mathbf{v}
\end{bmatrix},
\]

(1)

where \(\mathbf{Y} \in \mathbb{C}^{N \times N}\), \(\mathbf{v} \in \mathbb{C}^{N \times 1}\), and \(y_{00}\) are formed based on the system topology and the \(\pi\)-equivalent circuit of the distribution lines (see, e.g., [20, Chapter 6] for additional details on distribution line modeling). Finally, \(V_0\) denotes the voltage at the secondary of the transformer/substation. A constant-power load model is utilized, and \(P_{l,n}\) and \(Q_{l,n}\) denote the real and reactive demands at node \(n \in \mathcal{N}\) [20]. The active and reactive powers flowing into the feeder at the substation are denoted as \(P_0\) and \(Q_0\).

Let \(\mathcal{G} \subseteq \mathcal{N}\) be a set of nodes where DERs are located, and denote by \(P_i\) and \(Q_i\) the real and reactive powers injected by the DER located at node \(i \in \mathcal{G}\). We denote as \(\mathcal{Y}_i \subset \mathbb{R}^2\) the set of possible setpoints \((P_i, Q_i)\) for DER \(i\); the set \(\mathcal{Y}_i \subset \mathbb{R}^2\) captures hardware and operational constraints of the DER \(i\), and it is assumed to be convex and compact. Some examples are provided next.

**Photovoltaic (PV) systems:** Let \(P^\text{av}_i\) denote the available real power from a PV system and let \(S_i\) be the rated apparent capacity. Then, the set \(\mathcal{Y}_i\) is given by:

\[
\mathcal{Y}_i = \{(P_i, Q_i): 0 \leq P_i \leq P^\text{av}_i, P_i^2 + Q_i^2 \leq S_i^2\}.
\]

The set \(\mathcal{Y}_i\) is time varying since \(P^\text{av}_i\) depends on underlying irradiance conditions (it can be obtained via, for example, forecasting algorithms). The set \(\mathcal{Y}_i\) can also be modified to account for power factor constraints.

\(^1\)Upper-case (lower-case) boldface letters will be used for matrices (column vectors); \((.)^*\) for transposition; \((.)^\dagger\) complex-conjugate transpose; and, \((.)^\ddagger\) complex-conjugate transposition. \(\mathbb{R}\{\}\) and \(\mathbb{R}\{\}^\dagger\) denote the real and imaginary parts of a complex number, respectively, and \(i := \sqrt{-1}\). \(|\cdot|\) denotes the absolute value of a number or the cardinality of a set. For a given \(N \times 1\) vector \(\mathbf{x} \in \mathbb{R}^N\), \(\|\mathbf{x}\|_2 := \sqrt{\mathbf{x}^H \mathbf{x}}\); diag(\(\mathbf{x}\)) returns a \(N \times N\) matrix with the elements of \(\mathbf{x}\) in its diagonal. Further, \(\text{proj}_\mathcal{Y}(\mathbf{x})\) denotes the projection of \(\mathbf{x}\) onto the convex set \(\mathcal{Y}\). Given a matrix \(\mathbf{X} \in \mathbb{R}^{N \times N}\), \(x_{m,n}\) denotes its \((m,n)\)-th entry, \(\mathbf{X}_{n,:}\) denotes the n-th row, and \(\|\mathbf{x}\|_2\) denotes the \(\ell_2\)-induced matrix norm. \(\nabla f(\mathbf{x})\) returns the gradient vector of \(f(\mathbf{x})\) with respect to \(\mathbf{x} \in \mathbb{R}^N\). Finally, \(\mathbf{1}_N\) denotes the \(N \times 1\) vector with all ones, and \(\mathbf{0}_N\) denotes the \(N \times 1\) vector with all zeros.
Energy storage systems: The set $\mathcal{Y}_i$ for an energy storage system is given by:

$$\mathcal{Y}_i = \{(P_i, Q_i): P_{i}^{\text{min}} \leq P_i \leq P_{i}^{\text{max}}, P_i^2 + Q_i^2 \leq S_i^2\}$$

for given limits $P_{i}^{\text{min}}, P_{i}^{\text{max}}$. These limits are updated during the operation of the battery based on the state of charge. For example, if the battery is fully charged, then $P_{i}^{\text{min}} < 0$ and $P_{i}^{\text{max}} = 0$.

Variable frequency drives: For DERs such as water pumps and supply fans of commercial HVAC systems, the set $\mathcal{Y}_i$ can be described as:

$$\mathcal{Y}_i = \{(P_i, Q_i): 0 \leq P_i \leq P_{i}^{\text{max}}, P_i^2 + Q_i^2 \leq S_i^2\}$$

for given constant parameters $P_{i}^{\text{max}}$ and $S_i$.

The operating region of small-scale diesel generators can be modeled using box constraints. For DERs with discrete levels of output powers (e.g., electric vehicle chargers with discrete charging levels), $\mathcal{Y}_i$ represents the convex envelope of the possible operating points; see e.g., [16].

B. Approximate models for the Power-flow Equations

Let $s_{\text{inj}} := [S_1, \ldots, S_N] \in \mathbb{C}^N$ collect the net powers injected at nodes $N$, where $S_i = P_i - P_{i,i} + j(Q_i - Q_{i,i})$ for $i \in \mathcal{G}$, and $S_i = -P_i + jQ_{i,i}$ for $i \in \mathcal{N} \setminus \mathcal{G}$. Then, using (1), the complex-power injections can be compactly written as

$$s_{\text{inj}} = \text{diag}(v) i^* = \text{diag}(v)(Y^*v^* + \mathbf{y}^*V_0^*)$$

Assume that node 1 is connected to the substation via a distribution line, and recall that the power entering the feeder is given by $S_{10} = V_0 P_{10}$, where $J_{10}$ is the current flowing on the distribution line (0, 1). Particularly, with $y_{01} \in \mathbb{C}$ denoting the admittance of line (0, 1), $y_{01}^{\text{sh}} \in \mathbb{C}$ any passive shunt elements connected to node 0, and $y_{01}^{\text{sh}}$ the shunt component of the line (0, 1), $J_{10}$ is given by $J_{10} = y_{01} V_0 - V_1$, with $y_0 = g_0 + j b_0 := y_{00}^{\text{sh}} + y_{01}^{\text{sh}}$. Thus, $S_0$ can be rewritten as

$$S_0 = |V_0|^2 (y_{00}^{\text{sh}} + y_{01}^{\text{sh}}) - V_0 (y_{01}^{\text{sh}} V_1^*)$$

Unfortunately, the nontrivial nonlinearities in (2) and (3) hinder the possibility of seeking analytical closed-form solutions to $v$, $P_0$, and $Q_0$ (as a function of the network topology and composition, power injections, and voltage $V_0$).

To facilitate the design and analysis of computationally tractable controllers that afford a real-time implementation, the proposed approach will therefore leverage pertinent linearization approaches for (2)–(3). Particularly, we will utilize

$$|v| \approx \mathbf{A}s_{\text{inj}} + \mathbf{B}s_{\text{inj}} + \mathbf{c},$$

$$\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} \approx \mathbf{M}s_{\text{inj}} + \mathbf{N}s_{\text{inj}} + \mathbf{o},$$

where $p_{\text{inj}} := \mathbb{R}\{s_{\text{inj}}\}$ and $q_{\text{inj}} := \mathbb{R}\{s_{\text{inj}}\}$. The model parameters $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{B} \in \mathbb{R}^{N \times N}$, $\mathbf{M} \in \mathbb{R}^{2 \times N}$, $\mathbf{N} \in \mathbb{R}^{2 \times N}$, $\mathbf{c} \in \mathbb{R}^N$, and $\mathbf{o} \in \mathbb{R}^2$ can be obtained using suitable linearization methods for the AC power-flow equations. For example, one can leverage the approximation method proposed in [21], the method based on a first-order linear manifold approximant described in [22], [23], or the so-called “LinDistFlow” approximation [24]. In the Appendix, we will provide an extension of the approach of [21] to derive a more general approximate model for voltage magnitudes and powers at the substation. When the network model is not known, regression methods can be utilized based on real-time measurements of $v$, $P_0$, $Q_0$, and $s_{\text{inj}}$ (see, e.g., the recursive least-squares method in [25]).

The approximate model (4)–(5) facilitates the design of computationally-affordable algorithms. However, power setpoints obtained from (4)–(5) may cause violations of electrical limits. Section III will then show how to leverage appropriate measurements to enforce electrical limits while enabling effective tracking of setpoints at the substation.

III. FEEDER AS A VIRTUAL POWER PLANT

A. Problem Formulation

Control actions are performed in a discrete-time fashion at time instants $\{t_k = k\tau\}_{k \in \mathbb{N}}$, where $\tau > 0$ is the time required to compute one closed-loop iteration of the control strategy illustrated in Fig. 1. The value of $\tau$ is limited by communication delays involved in collecting measurements of voltages and powers at the substation and when broadcasting the signal $d$ to the individual DERs. Typically, $\tau$ can be on the order of subseconds to seconds [16], [18]. We start by formalizing a time-varying optimization problem to model operational objectives and constraints at each time instant $t_k$.

To this end, define the following quantities related to voltage magnitudes and power at the substation (derived from (4) and (5)):

$$g_{\text{inj}}^k(p, q) := V_0^\min - 2\tau_n^k - \sum_{i \in \mathcal{G}} (a_{i,i}^k P_i + b_{i,i}^k Q_i),$$

$$g_n^k(p, q) := \sum_{i \in \mathcal{G}} (a_{i,i}^k P_i + b_{i,i}^k Q_i) + \tau_n^k - V_0^\max,$$

$$P_0^k(p, q) := \sum_{i \in \mathcal{G}} (m_{i,i}^k P_i + n_{i,i}^k Q_i) + \tau_1^k,$$

where it follows from (4) and (5) that:

$$\sigma_n^k := c_n^k - \sum_{i \in \mathcal{N}} (a_{i,i}^k P_i^k + b_{i,i}^k Q_i^k),$$

$$\sigma_1^k := c_1^k - \sum_{i \in \mathcal{N}} (m_{i,i}^k P_i^k + n_{i,i}^k Q_i^k).$$

Let $P_{0,\text{set}}^k$ be the setpoint specified for the active power at the substation at time $t_k$ (with a positive sign representing power flowing into the feeder) and assume that the active power at the feeder head must track $P_{0,\text{set}}^k$ within a given tracking error $E_{0,k}^k > 0$; that is,

$$h_{0,k}^k |P_{0,\text{set}}^k(p, q) - P_{0,\text{set}}^k| \leq E_{0,k}^k,$$
where \( h^t_k = 1 \) if the feeder is requested to follow the setpoint \( P_{0, \text{set}}^t \) and \( h^t_k = 0 \) otherwise. With these preliminaries in place, consider the following optimization problem:

\[
\begin{align}
(P1^k) \quad & \min_{\mathbf{p}, \mathbf{q}} \sum_{i \in G} f^t_k(P_i, Q_i) \\
& \text{subject to } P_i, Q_i \in \mathcal{Y}^t_k, \quad \forall i \in G \quad (9a) \\
& \quad h^t_k(P^t_k(\mathbf{p}, \mathbf{q}) - P^t_0, \mathbf{q}) - P^t_{0, \text{set}} \leq E^t_k, \quad (9b) \\
& \quad - h^t_k(P^t_0(\mathbf{p}, \mathbf{q}) - P^t_{0, \text{set}}) \leq E^t_k, \quad (9c) \\
& \quad g^t_n(\mathbf{p}, \mathbf{q}) \leq 0, \quad \forall n \in M \quad (9d) \\
& \quad \tilde{g}^t_n(\mathbf{p}, \mathbf{q}) \leq 0, \quad \forall n \in M \quad (9f) \\
\end{align}
\]

where constraints (9e)–(9f) enforce voltage regulation (see (6a)–(6b)) at the measurements of the voltage magnitudes can be obtained. Functions \( f^t_k(\cdot) \) capture the DER output-power setpoints at the substation while enforcing voltage regulation. Deriving the optimal DER output-powers is not feasible in practice. In fact, i) due to underlying computational and communication limits, the algorithm enables effective tracking of approximations of linear models, its solution may lead to voltage violations and poor tracking performance.

Problem (9) defines a sequence of time-varying optimal DER output-power setpoints \( \{\mathbf{p}^{\text{opt}, t_k}, \mathbf{q}^{\text{opt}, t_k}\}_{k \in \mathcal{N}} \). However, solving problem (9) at each time \( t_k \) in a batch fashion is not feasible in practice. In fact, i) due to underlying computational and communication limits, the algorithm enables effective tracking of approximate linear models, its solution may lead to voltage violations and poor tracking performance.

In the next section, we will develop a computationally affordable online algorithm that continuously pursues the optimal solution trajectory of (9). With the aid of appropriate measurements, the algorithm enables effective tracking of setpoints at the substation while enforcing voltage regulation.

**B. Proposed Algorithm**

To begin the algorithm design, we first establish a few pertinent technical assumptions regarding (9):

**Assumption 1.** Function \( f^t_k(P_i, Q_i) \) is convex and continuously differentiable for each \( i \in G \) and for each \( t_k \). Define further the gradient map \( f^t_k(\mathbf{p}, \mathbf{q}) := [\nabla^T_{P_i} f^t_k(P_i, Q_i) \ldots \nabla^T_{P_n} f^t_k(P_n, Q_n), f^t_k(P_0, Q_0)]^T \).

Then, \( f^t_k : \mathbb{R}^{2N_0} \to \mathbb{R}^{2N_0} \) is Lipschitz continuous with constant \( L \) over \( \mathcal{Y}^t_k := \mathcal{Y}^t_{x_1} \times \ldots \times \mathcal{Y}^t_{x_N} \) for all \( t_k \).

**Assumption 2.** For all \( t_k \geq 0 \), there exist \( \{P_i, Q_i \in \mathcal{Y}^t_k\}_{i \in G} \) such that constraints (9c)–(9f) can be satisfied.

Since (9c)–(9f) are linear in \( \mathbf{p}, \mathbf{q} \), Assumption 2 implies that Slater’s condition holds. If equality constraints are included in \( (P1^k) \), then Assumption 2 must be properly modified to presuppose the existence of a strictly feasible solution.

The proposed algorithm leverages primal-dual methods applied to regularized Lagrangian functions [3], [4], [18]. Let \( \gamma^t := [\gamma^t_1, \ldots, \gamma^t_M]^T \) and \( \mu^t := [\mu^t_1, \ldots, \mu^t_M]^T \) collect the dual variables associated with (9e) and (9f), respectively; similarly, let \( \lambda^t \) and \( \zeta^t \) be the Lagrange multipliers associated with the constraints (9c)–(9d). With \( \mathbf{d} := \)
Based on measurements (m1)–(m3), the sequential execution of the following steps defines the proposed algorithm:

Real-time Virtual-power-plant Regulation

[S1a] Collect voltage-magnitude measurements \( \{ |V_n|^2 \}_{n \in \mathbb{N}} \).

[S1b] Collect measurement of \( P_{0}^t \).

[S2a] For all \( n \in \mathbb{N} \), update \( \gamma_{n+1}^k \) and \( \mu_{n+1}^k \) as follows:

\[
\gamma_{n+1}^k = \text{proj}_{\mathbb{R}^+} \{ \gamma_n^k + \alpha (V_{\min} - |V_n|^2 - c_{n}^k) \} \quad (12a)
\]

\[
\mu_{n+1}^k = \text{proj}_{\mathbb{R}^+} \{ \mu_n^k + \alpha (|V_n|^2 - V_{\max} - c_{n}^k) \} \quad (12b)
\]

[S2b] For the feeder head, if \( h^t_k \) is 1 update dual variables as follows:

\[
\lambda_{n+1}^k = \text{proj}_{\mathbb{R}^+} \{ \lambda_n^k + \alpha (P_{0}^t - P_{0, set}^t - E^t - \lambda_n^k) \} \quad (12c)
\]

\[
\zeta_{n+1}^k = \text{proj}_{\mathbb{R}^+} \{ \zeta_n^k + \alpha (P_{0, set}^t - P_{0}^t - E^t - c_n^k) \} \quad (12d)
\]

[S3a] Measure output powers \( P_{i}^t, Q_{i}^t \) at each DER \( i \in \mathcal{G} \).

[S3b] Update power setpoints at each DER \( i \in \mathcal{G} \) as:

\[
\begin{align*}
\mathbf{P}_{k+1}^t &= \text{proj}_{\mathbb{R}^+} \left\{ \mathbf{P}_{n+1}^t \right\} \\
\mathbf{Q}_{k+1}^t &= \text{proj}_{\mathbb{R}^+} \left\{ \mathbf{Q}_{n+1}^t \right\}
\end{align*}
\]

[S3c] Dispatch setpoints to each DER \( i \), and return to [S1a].

Steps [S1]–[S3] are performed at each time \( t \). The stepsize \( \alpha \) is a design parameter chosen as explained in Section III-C. The steps in (12) can be implemented in one of two ways:

- **Centralized implementation**: Steps [S1]–[S3] are implemented centrally at the utility/aggregator. The utility/aggregator collects measurements of voltages, DER output powers, and \( P_{0}^t \), executes steps (12), and relays the power setpoints \( P_{k+1}^t, Q_{k+1}^t \) to each DER \( i \in \mathcal{G} \).

- **Distributed implementation**: As illustrated in Fig. 2, steps [S1]–[S2] are performed at the utility/aggregator, while step [S3] is implemented locally at individual DERs. The utility/aggregator collects measurements of voltages and \( P_{0}^t \), computes \( \mathbf{d}^t_k \), and subsequently broadcasts \( \mathbf{d}^t_k \). Each DER updates \( P_{i}^t, Q_{i}^t \) based on \( \mathbf{d}^t_k \) and the (local) measurements \( P_{i}^t, Q_{i}^t \).

The setpoints for the power at the feeder head \( P_{0, set}^t \) and the target accuracy \( E^t \) are specified (and continuously updated) by the Independent System Operator (ISO) based on transmission-system operating requirements and ISO-utility market agreements.

The ability of the updates to track the optimizers \( \mathbf{z}_{n_1}^t = \{ \mathbf{p}^t, \mathbf{q}^t \} \) of (11) will be analytically established and numerically verified next.

C. Convergence of Algorithm

In this section, convergence of the updates in (12) is established. Begin by noticing that there exists a constant \( G_\gamma \) such that

\[ \| \nabla_{[p, q]} g_i^t(p, q) \|_2 \leq G_\gamma \] and

\[ \| \nabla_{[p, q]} g_i^t(p, q) \|_2 \leq G_\gamma \]

for all \( p, q \in \mathcal{Y}^k \) and for all \( t_k \), where \( g_i^t(p, q) \in \mathbb{R}^M \) and \( g_i^t(p, q) \in \mathbb{R}^M \) are vectors stacking all the functions \( g_i^t(p, q) \), \( n \in \mathbb{N} \), and \( g_i^t(p, q) \), \( n \in \mathbb{N} \), respectively. Also, define the scalar \( G_\eta \) such that

\[ \| \nabla_{[p, q]} f_i^0(p, q) \|_2 \leq G_\eta \]

for all \( p, q \in \mathcal{Y}^k \) and \( t_k \). Then, define the constant \( G := \max\{G_\gamma, G_\eta\} \). For example, since functions \( g_i^t(p, q) \) and \( \mathbf{g}(p, q) \) are linear [c.f. (4)], a possible bound \( G_\gamma \) could be

\[ G_\gamma = \max \{ \| A \|_2, \| M_1 \|_1, \| N_1 \|_1 \} \|_2 \]

and thus, \( G = \max \{ \| A \|_2, \| M_1 \|_1, \| N_1 \|_1 \} \|_2 \} \) (we recall that \( M_1 \) and \( N_1 \) denote the row 1 of matrices \( M \) and \( N \), respectively). This constant will be used next in the convergence result for the proposed algorithm.

With respect to the measurements (m1)–(m3), the following are assumed:

**Assumption 3.** There exists a scalar \( 0 \leq \epsilon_p < +\infty \) such that:

\[ \left\| \begin{bmatrix} \mathbf{p}^* \backslash \mathbf{q}^* \end{bmatrix} - \begin{bmatrix} \mathbf{p}^*_t \backslash \mathbf{q}^*_t \end{bmatrix} \right\|_2 \leq \epsilon_p \]

for all \( t_k, k \in \mathbb{N} \).

**Assumption 4.** There exist constants \( 0 \leq \epsilon_v < +\infty \) and \( 0 < \epsilon_s < +\infty \) such that:

\[ \| (A^t \mathbf{p}^*_{0, set} + B^t \mathbf{q}^*_{0, set} + \mathbf{a}^t) - \mathbf{v}^t \|_2 \leq \epsilon_v \]

\[ \| (M_1, \mathbf{p}^*_{0, set} + N_1 + \mathbf{q}^*_{0, set} + \mathbf{a}^t) - \mathbf{P}^*_0 \|_2 \leq \epsilon_s \]

for all \( t_k, k \in \mathbb{N} \).

**Assumption 3** provides a bound for the discrepancy between the commanded setpoint \( \mathbf{P}^*_0, \mathbf{Q}^*_0 \) and the actual output powers of each DER \( i \) [16], [17], [26]. Particularly, the output powers may not coincide with the commanded setpoints \( P_{i}^t, Q_{i}^t \) because: i) measurements may be prone to error, ii) the settling time of the DER output-powers may be larger than \( \tau \) [17], or iii) \( \gamma^t \) may represent only an estimate of the actual operating region [16]. **Assumption 4** accounts for measurement and linearization errors. Steps in (12) represent a modified online primal-dual-gradient method applied to the saddle-point problem (11) where actual measurements from the distribution system replace the mathematical models for voltages and powers in (9c)–(9f). When using measurements (m1)–(m3) in (12), the optimal and dual updates involve inexact gradient steps [27]. Based on this observation, the error in the setpoint computation is bounded in the following lemma.

**Lemma 1:** Let \( \{ \mathbf{p}_{k+1}^t, \mathbf{q}_{k+1}^t \} \) be the (exact) primal iterates given by replacing \( \{ \mathbf{p}^t_k, \mathbf{q}^t_k \} \) with \( \{ \mathbf{p}^t_k, \mathbf{q}^t_k \} \) in (12e). Suppose that **Assumptions 1 and 3** hold. Then, whenever \( \{ \mathbf{p}_{k+1}^t, \mathbf{q}_{k+1}^t \} \neq \{ \mathbf{p}^t_k, \mathbf{q}^t_k \} \), the error in the gradient in (12e) is such that:

\[ \left\| \begin{bmatrix} \mathbf{p}^t_k \backslash \mathbf{q}^t_k \end{bmatrix} - \begin{bmatrix} \mathbf{p}_{k+1}^t \backslash \mathbf{q}_{k+1}^t \end{bmatrix} \right\|_2 \leq \alpha(L + \nu)\epsilon_p \]

where \( L \) is the Lipschitz constant of the gradient map \( f_i^t(p, q) \).

The main convergence result is established next.

**Theorem 1:** Consider the sequence \( \{ \mathbf{z}_{k+1}^t \} := \{ \mathbf{p}^t_k, \mathbf{q}^t_k, \mathbf{d}^t_k \} \) generated by (12). Let **Assumptions 1–4** hold and suppose that, for fixed positive scalars \( \epsilon, \nu > 0 \), the stepsize \( \alpha > 0 \) is chosen such that

\[ \rho(\alpha) := \sqrt{1 - 2\alpha \min\{\nu, \epsilon\} + \alpha^2 B} < 1 \]
where $B = (L + \nu + 4G)^2 + 4(G + \epsilon)^2$. Then the sequence \( \{z^{t_k}\} \) converges Q-linearly to \( z^{*t_k} = (p^{*t_k}, q^{*t_k}, d^{*t_k}) \) up to the asymptotic error bound given by:

$$
\limsup_{t_k \to \infty} \|z^{t_k} - z^{*t_k}\|_2 = \frac{1}{1 - \rho(\alpha)} (\alpha \epsilon + \sigma) \quad (18)
$$

where \( \epsilon := \sqrt{(L + \nu)^2 e_p^2 + 2e_c^2 + 2e_o^2} \) and \( \sigma \geq 0 \) a given constant such that \( \|z^{t_k+1} - z^{*t_k}\| \leq \sigma \) for all \( t_k \geq 0 \). □

The result (18) bounds the maximum discrepancy between the DER setpoints generated by the algorithm (12) and the time-varying optimizer of (11) at any time \( t_k \). The bound (18) is a function of the parameters \( \alpha, \epsilon, \) and \( \nu \) as well as the measurement and linearization errors. The parameters \( \alpha, \epsilon, \) and \( \nu \) can be chosen to satisfy condition

$$
0 < \alpha < 2 \min (\epsilon, \nu) \quad (B) = \frac{\min (\epsilon, \nu)}{(L + \nu + 4G)^2 + 4(G + \epsilon)^2}
$$

to achieve Q-linear convergence.

The bound (18) depends on the underlying dynamics of the distribution system through \( \sigma \). Particularly, \( \sigma \) captures the maximum difference between the solutions of (11) at two consecutive time instants \( t_k \) and \( t_{k+1} \) [4]; in the current problem formulation, variations in \( z^{*t_k} \) are due to changes in the powers injected/consumed by non-controllable assets [cf. (7)], time-varying setpoints \( F_{0,\text{set}} \), possibly time-varying cost functions \( \{f_{i}^{t_k}(P, Q)\} \), and variations in the voltage limits When \( z^{*t_k} \) varies slowly in time, bound (18) becomes tighter. Expression (18) provides an asymptotic bound for the tracking error; in the Appendix, we discuss how to obtain an upper bound on the tracking error at each iteration. The result (18) can also be interpreted as input-to-state stability, where the optimal trajectory \( \{z^{*t_k}\} \) of the time-varying problem (9) is taken as a reference. Finally, notice that when \( \epsilon = 0 \) and \( \sigma = 0 \), the algorithm converges to the solution of the static optimization problem (11).

The proof of Theorem 1 is provided in the Appendix. Although (18) is related to the time-varying solution of the linearized problem (9), ongoing efforts are looking at establishing similar bounds against the time-varying solution of the nonconvex nonlinear counterpart to (9).

**Remark 1 (local DER controller)** The algorithm (12) produces setpoints \( \{P_n^{t_k}, Q_n^{t_k}\} \in \mathcal{Y}_n^{t_k} \) for the output powers of the DERs. It is assumed that the DERs are endowed with controllers that are designed so that, upon receiving the setpoint, the output powers are driven to the commanded setpoints. Relevant dynamical models for the output powers of inverters operating in a grid-connected mode are discussed in e.g., [28], [29] and can be found in datasheets of commercially available DERs. Assumption 3 accounts for measurement errors and bounds the discrepancy between the commanded setpoint and the actual output power when updates of the setpoints may be performed faster than the DERs’ settling times; Assumption 3 is valid, for example, when the DER’s response to a step-change in the setpoint follows a first-order model [28], [29].

**Remark 2 (feasibility of powers at the substation)** Assumption 2 implies that the setpoints for the active power at the substation are feasible. The set of feasible setpoints for the active and reactive power at the substation can be assessed (and optimized) by solving suitable optimization problems at a slower time scale. See, for example, the multi-period optimization problem proposed in [30]. The feasible setpoints \( F_{0,\text{set}} \) can be computed based on the operating regions for DERs \( \mathcal{Y}_n^{t_k} \) as well as given operational limits. An alternative approach to assess the flexibility of agglomerations of DERs is presented in [31].

**IV. NUMERICAL EXPERIMENTS**

**A. Test Case 1**

Consider a modified version of the IEEE 37-node test feeder shown in Fig. 3. The modified network is obtained by considering a single-phase equivalent, and by replacing the loads on phase “c” specified in the original dataset with real load data measured from feeders in a neighborhood called Anatolia in California during a week in August 2012 [32]. Time-series data for the non-controllable loads have a granularity of 1 second and are plotted in Fig. 4. Line impedances and shunt admittances are adopted from the original dataset. With reference to Fig. 3, it is assumed that eighteen PV systems are located at nodes 4, 7, 10, 13, 17, 20, 22, 23, 26, 28, 29, 30, 31, 32, 33, 34, 35, and 36. The rating of these inverters are 300 kVA for \( i = 3, 350 \) kVA for \( i = 15, 16, \) and 200 kVA for the remaining ones. The generation profiles are simulated based on the real solar irradiance data available in [32] and have a granularity of 1 second. As an instance, the power available from a PV system with capacity 50 kW is reported in Fig. 4. The dynamics of the output powers of the inverters are modeled using a first-order system; different values for the time constant of the first-order system will be considered throughout this section. Energy storage systems are placed at nodes 3 and 25, their maximum state of charge is 200 kWh, capacity is 50 kVA, and charging and discharging efficiencies are set to 90%. With these simulation settings, overvoltage conditions would occur if the PV inverters operate according to business-as-usual practices (that is, unity power factor and maximum power point). This give us the opportunity to corroborate the ability of the proposed algorithm to track setpoints for the active power at the substation, while concurrently enforcing voltage regulation. The voltage limits \( V_{\text{max}} \) and \( V_{\text{min}} \) are set to 1.05 pu and 0.95 pu, respectively.

For the controllers illustrated in Fig. 2, the parameters are set as \( \nu = 10^{-3}, \epsilon = 10^{-4}, \) and \( \alpha = 0.1. \) The target optimization objective (9a) is set to \( f_{0,\text{set}}^{t_k}(P_n, Q_n) = c_p(P_{\text{av},n} - P_n^{t_k})^2 + c_q(Q_n^{t_k})^2 \) for PV systems (with \( P_{\text{av},n} \) denoting the maximum real power available at PV system \( n \)) and \( f_{0,\text{set}}^{t_k}(P_n, Q_n) = c_p(P_n^{t_k})^2 + c_q(Q_n^{t_k})^2 \) for batteries. The coefficients are set to \( c_p = 3, c_q = 1 \) for the PV systems and \( c_p = 1, c_q = 1 \) for the batteries. With these functions, the PV systems minimize the power curtailment while the batteries minimize the deviation from a prescribed charging/discharging profile. Functions \( \{f_{i}^{t_k}\} \), however, could accommodate a variety of alternative performance objectives, including the rewards from ancillary service provisioning [33], [34] (to be maximized). The only requirement for each func-
The tracking performance of the controllers (12) is compared with the strategy (19) in Fig. 7. The coefficients $\gamma_n$ are set as $\gamma_n = \gamma/|G|$ for a prescribed $\gamma$. When $\gamma \leq 1$, (19) successfully tracks the setpoints $P_{0,\text{set}}^{\text{th}}$ and may converge to the new setpoint faster than the proposed controller when $\gamma = 1$. On the other hand, when $\gamma > 1$, (19) exhibits overshoot behavior that prevents the feeder from following ramping commands. This implies that to set $\gamma$ properly, one requires knowledge of the number of participating DERs; on the other hand, the stepsize $\alpha$ in (12) depends only on given (and fixed) problem parameters. It is also worth reiterating that the strategy (19) does not address the voltage regulation problem and does not minimize functions $\{f_{ik}(P_n, Q_n)\}$ as demonstrated next.

Figure 8 illustrates the voltage profiles obtained with the proposed controllers. The profiles are compared with the case where the controller (19) is complemented with a local Volt/Var rule and with the case where the proposed algorithm is implemented in a network-agnostic (NA) fashion; that is, voltage constraints are discarded, $M = -I$, $N = 0$, and...
Fig. 8. Voltage profile $|V_n^a|$ in pu achieved by the proposed controllers.

Fig. 9. Cost $\sum_n f_n^{kW}(P_n, Q_n)$ achieved by (12) and (19).

Fig. 10. Performance of the proposed control scheme for different values of $\tau$ and time constants of the inverters.

$\mathbf{0} = 0$. It can be seen that the proposed controllers enforce voltage regulation, and a flat voltage profile is obtained; on the other hand, the solution (19) + Volt/VAr may not confine voltages within bounds. The NA implementation exhibits similar tracking performances but leads to voltage violations. Figure 9 illustrates the cost $\sum_n f_n^{kW}(P_n, Q_n)$ achieved by the proposed solution. It can be seen that, by encapsulating optimization objectives, the proposed method systematically achieves lower operational costs compared to the heuristic (19) + Volt/VAr.

Finally, Fig. 10 demonstrates the tracking capabilities of (12) for different values of $\tau$ and inverter time constants. We see that the proposed method is resilient to slow-responding DERs, and ensures tracking accuracy even when the dynamics of the inverters are on the order of 1 s. On the other hand, the tracking performance deteriorates when the time $\tau$ required to perform one closed-loop iteration in Fig. 2 increases.

In this test case, the average computational time for step (12e) was 0.17 ms on a MacBook with a 3.1 GHz Intel Core i7 and 16 GB 1867 MHz DDR3. Similar computational times were obtained for the update of the dual variables.

B. Test Case 2

We now consider a test case with a mix of residential-scale PV inverters with capacities of 3kVA and 5kVA, and commercial-scale PV installations with capacities of 10kVA, 50kVA and 100kVA. The number of PV inverters per node along with their aggregate capacity is summarized in Table I. We also consider three utility-scale energy storage systems, as summarized in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th>PV system</th>
<th>Node</th>
<th>Units</th>
<th>Total Capacity [kVA]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>23</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>26</td>
<td>4</td>
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<td>20</td>
</tr>
<tr>
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</tr>
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<td>2</td>
<td>20</td>
</tr>
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<td>2</td>
<td>20</td>
</tr>
<tr>
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<td>34</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>13</td>
<td>35</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>14</td>
<td>36</td>
<td>3</td>
<td>300</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy storage systems</th>
<th>Node</th>
<th>Units</th>
<th>Total Capacity [kWh]</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>3</td>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>3</td>
<td>200</td>
</tr>
</tbody>
</table>

Fig. 11 shows the tracking performance of proposed algorithm (12). The performance is compared with the strategy (19) when the coefficients $\gamma_n$ are set as $\gamma_n = \gamma / |G|$, with $\gamma = 1$. Both algorithms exhibit good tracking capabilities, with a relative error of 2.25% and 2.1%. However, (19) leads to instability when $\gamma = 2$ (and it is not plotted for ease of readability).
Fig. 12. Comparison with an offline optimization solution where power setpoints are commanded to the DERs only upon convergence of the algorithm; different convergence times are simulated.

Fig. 13. Performance of the proposed algorithm when a setpoint for the active power at the substation is not feasible. The setpoint $P_{0, \text{set}}^t = -950$ kW given between 12:10 and 12:15 is not feasible.

Fig. 14. Performance of the proposed control scheme for different values of $\tau$ and time constants of the inverters in Test Case II.

DERs’ output powers.

Finally, Fig. 14 demonstrates the tracking capabilities of (12) for different values of $\tau$ and inverter time constants. Similar to the results obtained in Test Case I, the proposed method is resilient to slow-responding DERs. On the other hand, for higher values of $\tau$, the controllers respond in a slower manner to variations in the non-controllable loads and irradiance.

To further validate the performance gains with respect to an NA implementation, we consider an additional test case where distribution lines feature shunts elements; in the IEEE 37-node test feeder, this introduces a more pronounced diversification of the coefficients in the matrix $M$ in (5). The tracking error of the proposed method turned out to be 1.8%, whereas the NA implementation exhibited a tracking error of 6.9%. The tracking error was computed by taking the time average of $|P_0^t - P_{0, \text{set}}^t|/P_{0, \text{set}}^t$. Overall, the proposed network-cognizant implementation allows one to tightly control voltages within limits while achieving higher tracking performance.

V. Concluding Remarks

This paper developed an algorithmic framework to enable distribution networks to emulate a virtual power plants that respond to regulation requests received from the transmission system. The controllers adjust the output powers of individual DERs in response to setpoints for the power at the feeder head, while concurrently regulating voltages within the feeder and maximizing customers’ and utility’s performance objectives. With respect to stability and tracking capabilities, analytical results were presented. Numerical experiments corroborated the analytical findings and assessed the tracking performance for different speeds of updates for the DER’s commanded powers. It is shown that the proposed method is resilient to slow-responding DERs, and ensures tracking accuracy even when the dynamics of the inverters are on the order of seconds. It is also shown that the proposed approach outperforms traditional offline optimization approaches.

Future research endeavors will broaden the applicability of the proposed algorithm to account for DERs with discrete power commands (including on/off decisions), and will look at the development of real-time algorithm for nonconvex problem formulations. Finally, notice that the proposed real-time algorithm is in fact a myopic control strategy. We will...
then pursue the development of online algorithms for time-varying multi-period optimization problems.

**APPENDIX**

A. Linear approximation

The parameters of the AC power-flow approximations (4)–(5) can be computed (and periodically updated) in multiple ways. Suitable linearization methods for the AC power-flow equations, such as the methods outlined in [21], [23] and the so-called LinDistFlow approximation [24] can be utilized. Alternatively, the model parameters can be estimated via regression-based methods such as the online RLS algorithm, based on real-time measurements of voltages and powers flows. In this section, an example for a linear approximation method of the AC power-flow equations is outlined; particularly, we broaden the approach of [21] to provide a more general approximation of voltage magnitudes and derive an approximate relationship between powers at the substation and net injected powers throughout the feeder. For ease of exposition, the temporal index $t_k$ is dropped.

Central to the linearization approach is to express the voltages $v$ as $v = v_{\text{nom}} + v_e$, where $v_{\text{nom}}$ is some nominal-voltage vector (i.e., the linearization point) determined a priori, and entries of $v_e$ capture perturbations around $v_{\text{nom}}$. With $v_{\text{nom}}$ appropriately determined, we need to solve for $v_e$ that satisfies the following equation:

$$s = \text{diag}(v_{\text{nom}} + v_e) (Y^*(v_{\text{nom}} + v_e)^* + \mathcal{F}^* V_0^*)^{-1}$$  \hspace{1cm} (20)

Expanding (20), one gets

$$s = \text{diag}(v_{\text{nom}}) Y^* v_{\text{nom}}^* + \text{diag}(v_{\text{nom}}) Y^* v_e^* + \text{diag}(v_e) Y^* v_{\text{nom}}^* + \text{diag}(v_e) Y^* v_e^* + \text{diag}(v_{\text{nom}}) \mathcal{F}^* V_0^* + \text{diag}(v_e) \mathcal{F}^* V_0^*.$$  \hspace{1cm} (21)

Neglecting the second-order term $\text{diag}(v_e) Y^* v_e^*$, and recognizing that

$$\text{diag}(v_e) Y^* v_{\text{nom}}^* = \text{diag}(Y^* v_{\text{nom}}) v_e, \quad \text{diag}(v_e) \mathcal{F}^* V_0^* = V_0^* \text{diag}(\mathcal{F}^*) v_e,$$  \hspace{1cm} (22)

and reorganizing terms, (21) can be compactly rewritten as

$$\Gamma v_e + \Xi v_e^* = s - s_{\text{nom}},$$  \hspace{1cm} (23)

where $\Gamma \in \mathbb{C}^{N \times N}$, $\Xi \in \mathbb{C}^{N \times N}$, and $s_{\text{nom}} \in \mathbb{C}^N$ are given by

$$\Gamma := \text{diag}(Y^* v_{\text{nom}}^* + \mathcal{F}^* V_0^*), \quad \Xi := \text{diag}(v_{\text{nom}}) Y^*,$$  \hspace{1cm} (24, 25)

$$s_{\text{nom}} := \text{diag}(v_{\text{nom}})(Y^* v_{\text{nom}}^* + \mathcal{F}^* V_0^*).$$  \hspace{1cm} (26)

The next step consists solving for the voltage perturbation vector $v_e$, using which one can recover an approximation to the actual solution $v$. Thus, decomposing all quantities in (23) into their real and imaginary parts, one can solve for $\mathbb{R}\{v_e\}$ and $\mathbb{I}\{v_e\}$ (and hence, for $v_e$) from

$$\begin{bmatrix} \mathbb{R}\{v_e\} \\ \mathbb{I}\{v_e\} \end{bmatrix} = \mathbf{H} \begin{bmatrix} p_{\text{inj}} \\ q_{\text{inj}} \end{bmatrix} - \mathbf{H} \begin{bmatrix} p_{\text{nom}} \\ q_{\text{nom}} \end{bmatrix},$$  \hspace{1cm} (27)

where $p_{\text{nom}} := \mathbb{R}\{s_{\text{nom}}\}$, $q_{\text{nom}} := \mathbb{I}\{s_{\text{nom}}\} \in \mathbb{R}^N$ denote the active- and reactive-power injected into the network at the nominal voltage, $v_{\text{nom}}$, and $\mathbf{H} \in \mathbb{R}^{2N \times 2N}$ is defined as follows:

$$\mathbf{H} := \begin{bmatrix} \mathbb{R}\{\Gamma\} + \mathbb{R}\{\Xi\} & -\mathbb{I}\{\Gamma\} + \mathbb{I}\{\Xi\} \\ \mathbb{I}\{\Gamma\} + \mathbb{I}\{\Xi\} & \mathbb{R}\{\Gamma\} - \mathbb{R}\{\Xi\} \end{bmatrix}^{-1}.$$  \hspace{1cm} (28)

To aid subsequent discussions, we will find it useful to denote the $N \times N$ blocks of $\mathbf{H}$ composed of by $\mathbf{H}^{(11)}$, $\mathbf{H}^{(12)}$, $\mathbf{H}^{(21)}$, and $\mathbf{H}^{(22)}$. In particular, this allows one to express

$$\mathbb{R}\{v_e\} = \mathbf{H}^{(11)} p_{\text{inj}} + \mathbf{H}^{(12)} q_{\text{inj}} + h_e,$$  \hspace{1cm} (29a)

$$\mathbb{I}\{v_e\} = \mathbf{H}^{(21)} p_{\text{inj}} + \mathbf{H}^{(22)} q_{\text{inj}} + h_i,$$  \hspace{1cm} (29b)

where $h_e := -\mathbf{H}^{(11)} p_{\text{nom}} - \mathbf{H}^{(12)} q_{\text{nom}}$ and $h_i := -\mathbf{H}^{(21)} p_{\text{nom}} - \mathbf{H}^{(22)} q_{\text{nom}}$.

Next, we leverage (27) to obtain (4). Begin by expressing:

$$v = v_{\text{nom}} + v_e = \text{diag}(e^{\theta_{\text{nom}}})[v_{\text{nom}} + v_e].$$  \hspace{1cm} (30)

Then, left multiplying (30) by $\text{diag}(e^{-j\theta_{\text{nom}}})$ we obtain:

$$\text{diag}(e^{-j\theta_{\text{nom}}})v = [v_{\text{nom}} + \text{diag}(e^{-j\theta_{\text{nom}}})v_e] = \text{diag}([v_{\text{nom}}]) \left(1_N + \text{diag} (e^{-j\theta_{\text{nom}}}) \text{diag} ([v_{\text{nom}}])^{-1} v_e \right).$$

Considering the (element-wise) magnitude on both sides above, it follows that:

$$|v| = \text{diag}([v_{\text{nom}}]) 1_N + \text{diag} (e^{-j\theta_{\text{nom}}}) \text{diag} ([v_{\text{nom}}])^{-1} v_e.$$  \hspace{1cm} (32)

Consider the approximation $1_N + \nu \approx 1_N + \Re(\nu)$ for $|\nu| \ll 1_N$ (element-wise) where $\nu \in \mathbb{C}^N$. Since $v_e$ represents a small perturbation around $v_{\text{nom}}$, it is reasonable to assume that

$$\text{diag}(e^{-j\theta_{\text{nom}}}) \text{diag} ([v_{\text{nom}}])^{-1} v_e \ll 1_N.$$  \hspace{1cm} (33)

Therefore, from (32), it follows that $|v| \approx |v_{\text{nom}}| + \Re \left(\text{diag}(e^{-j\theta_{\text{nom}}}) v_e\right)$ and, finally, from (27) one has that

$$|v| = |v_{\text{nom}}| - \Theta_{\text{nom}} H \begin{bmatrix} p_{\text{nom}} \\ q_{\text{nom}} \end{bmatrix} + \Theta_{\text{nom}} H \begin{bmatrix} p_{\text{inj}} \\ q_{\text{inj}} \end{bmatrix},$$  \hspace{1cm} (34)

where $H$ is defined in (28), and we define $\Theta_{\text{nom}} \in \mathbb{R}^{N \times 2N}$ as $\Theta_{\text{nom}} := [\text{diag}(\cos(\theta_{\text{nom}})) \text{diag}(\sin(\theta_{\text{nom}}))]$. Notice that (34) expresses the vector of node-voltage magnitudes as a linear function of the active- and reactive-power injections in the network. Equation (4) can be obtained upon setting $c = |v_{\text{nom}}| - \Theta_{\text{nom}} H [p_{\text{nom}} q_{\text{nom}}]^T$ and appropriately including the entries of matrix $\Theta_{\text{nom}} H$ into $A$ and $B$.

Approximation (5) is derived next. With $\{e_i \in \mathbb{R}^N\}_{i=1}^N$ denoting the vector basis for $\mathbb{R}^N$, it follows that $V_i$ can be rewritten as $V_i = e_i^T(v_{\text{nom}} + \Re\{v_e\} + j\Im\{v_e\})$ and, thus:

$$S_0 = |V_0|^2(y_{01} + y_0) - V_0^* y_{01} (e_i^T(v_{\text{nom}} + \Re\{v_e\} + j\Im\{v_e\}))^*.$$  \hspace{1cm} (35)

Substituting (29) in (35) and rearranging terms, the approximate linear relationship (5) between the power at the feeder
head \( S_0 = P_0 + j Q_0 \) and the net power injections \( p_{\text{inj}}, q_{\text{inj}} \) can be derived by setting \( M, N, \) and \( o \) as:

\[
\begin{bmatrix}
M \\
N \\
o
\end{bmatrix} = \begin{bmatrix}
-\psi_1 & 0 & \psi_2 & 0 \\
\psi_2 & 0 & \psi_1 & 0 \\
0 & -\psi_1 & 0 & \psi_2 \\
0 & \psi_2 & 0 & \psi_1
\end{bmatrix} \begin{bmatrix}
H^{(11)} \\
H^{(12)} \\
H^{(21)} \\
H^{(22)}
\end{bmatrix},
\]

(36)

where the following scalars have been defined for brevity:

\[
\begin{align*}
\psi_1 &= |V_0|^2 (\cos(\theta_0) g_{01} + \sin(\theta_0) b_{01}) \\
\psi_2 &= |V_0|^2 (\cos(\theta_0) g_{01} - \sin(\theta_0) b_{01})
\end{align*}
\]

(37)

(38)

It is worth pointing out that the linear model presented in this subsection can be extended to the multiphase unbalanced case. For example, the fixed-point methodologies presented in [35] can be utilized to derive approximate linear relationships of voltage magnitudes and powers at the substation in multiphase settings. Alternatively, the first-order Taylor method proposed in [22] can be utilized.

B. Proof of Lemma 1

Recall that the exact update of the primal variables can be obtained by replacing the power measurements \( \{p^{t_k}, q^{t_k}\} \) with the iterates \( \{p^{t_{k+1}}, q^{t_{k+1}}\} \) in (12e); that is,

\[
\begin{bmatrix}
p^{t_{k+1}} \\
qu^{t_{k+1}}
\end{bmatrix} = \text{proj}_{y^{t_k}} \left\{ \left\{ \begin{array}{c}
p^{t_k} \\
qu^{t_k}
\end{array} \right\} - \alpha \nabla_{\{p, q\}} L^{t_k}(p, q, d)|_{p^{t_k}, q^{t_k}, d^{t_k}} \right\}
\]

(39)

where \( \gamma^{t_k} = \gamma^{t_k} \times \ldots \times \gamma^{t_1} \) is the Cartesian product of the operating regions of the DERS. For brevity, let \( u^{t_k} := [(p^{t_k})^T, (q^{t_k})^T]^T \) and \( u^{t_k} := [(p^{t_k})^T, (q^{t_k})^T]^T \); further, collect the measured output powers in the vector \( \hat{u}^{t_k} := [(\gamma^{t_k})^T, (\mu^{t_k})^T, (\chi^{t_k})^T] \). Leveraging the non-expansive property of the projection operator, and using the bounds in Assumption 1 and 3, it follows that

\[
\begin{align*}
\|u^{t_k} - u^{t_{k+1}}\|_2 &\leq \alpha \left( L^{t_k}(u, d)|_{u^{t_k}, d^{t_k}} - L^{t_k}(u, d)|_{u^{t_{k+1}}, d^{t_{k+1}}} \right)_2 \\
&= \alpha \left( f^{t_k}(u)|_{u^{t_k}} - f^{t_k}(u)|_{u^{t_{k+1}}} + \nu(u^{t_k} - \hat{u}^{t_k}) \right)_2 \\
&\leq \alpha L \left( u^{t_k} - \hat{u}^{t_k} \right)_2 + \alpha \nu \left( u^{t_k} - \hat{u}^{t_k} \right)_2 \\
&\leq \alpha L \left( u^{t_k} - \hat{u}^{t_k} \right)_2 + \alpha \nu \left( u^{t_k} - \hat{u}^{t_k} \right)_2
\end{align*}
\]

(40a)

(40b)

(40c)

(40d)

(40e)

where the first term on the right hand side of (40d) follows from the Lipschitz continuity of the gradient map \( F^{t_k}(p, q) \). Step (40e) then follows from (13).
we used Hölder inequality to obtain (42b) from (42a). With $G = \max\{G_g, G_0\}$ and using Hölder inequality, it follows that

$$\|\Phi^{t_k}(z_1) - \Phi^{t_k}(z_2)\|_2 \leq \hat{B}\|z_1 - z_2\|_2.$$  \hspace{1cm} (43)

The results of Lemma 2 are utilized next to prove the bound in (18). To this end, define the time-varying operator $\Phi^{t_k}_e$ as:

$$\Phi^{t_k}_e : \{z^{t_k}\} \mapsto \begin{bmatrix} \nabla_{[p, q]}L^{t_k}(p, q, d)\big|_{p^{t_k}, q^{t_k}, d^{t_k}} - (\tilde{V}^{t_k} - V_{\max} - \epsilon_\mu^{t_k}) \\ (|\tilde{V}^{t_k}| - V_{\max} - \epsilon_\mu^{t_k}) \\ - (P^{t_k}_0 - P^{t_k}_{0, set} - E - \epsilon_\ell^{t_k}) \\ - (P^{t_k}_{0, set} - P^{t_k}_0 - E - \epsilon_\ell^{t_k}) \end{bmatrix},$$

where $\tilde{V}^{t_k}$ is a vector collecting measurements of the voltage collected at time $t_k$. Using $\Phi^{t_k}_e$, the steps of the algorithm can be compactly rewritten as

$$z^{t_k+1} = \text{proj}_{\mathbb{C}^{M \times R_{\mathbb{R}^d} \times R_{\mathbb{R}^d}}} \left\{ z^{t_k} - \alpha \Phi^{t_k}_e(z^{t_k}) \right\}. \hspace{1cm} (44)$$

By standard optimality conditions, the optimizer is a fixed point of the iterations (44), i.e., $z^{*, t_k-1} = \text{proj}_{\mathbb{C}^{M \times R_{\mathbb{R}^d} \times R_{\mathbb{R}^d}}} \left\{ z^{*, t_k-1} - \alpha \Phi^{t_k}_e(z^{*, t_k-1}) \right\}$. Consider then writing:

$$\|z^{t_k} - z^{*, t_k-1}\|_2 = \left\| \text{proj}_{\mathbb{C}^{M \times R_{\mathbb{R}^d} \times R_{\mathbb{R}^d}}} \left\{ z^{t_k-1} - \alpha \Phi^{t_k-1}_e(z^{t_k-1}) \right\} \right\|^2 - \left\| \text{proj}_{\mathbb{C}^{M \times R_{\mathbb{R}^d} \times R_{\mathbb{R}^d}}} \left\{ z^{*, t_k-1} - \alpha \Phi^{t_k-1}_e(z^{*, t_k-1}) \right\} \right\|^2 \hspace{1cm} (45)$$

and utilize the non-expansivity property of the projection operator to obtain

$$\|z^{t_k} - z^{*, t_k-1}\|_2 \leq \|z^{t_k-1} - \alpha \Phi^{t_k-1}_e(z^{t_k-1})\|_2 - \|z^{*, t_k-1} + \alpha \Phi^{t_k-1}_e(z^{*, t_k-1})\|_2. \hspace{1cm} (46)$$

By utilizing real measurements in the map $\Phi^{t_k-1}_e(z^{t_k-1})$, it follows that:

$$\Phi^{t_k-1}_e(z^{t_k-1}) - \Phi^{t_k-1}_e(z^{*, t_k-1}) = e^{t_k}. \hspace{1cm} (47)$$

where the vector $e^{t_k}$ captures measurement errors as well as model mismatches, and it is defined as:

$$e^{t_k} := \begin{bmatrix} \nabla_{[p, q]}L^{t_k}(p^{t_k}, q^{t_k}, d^{t_k}) - \nabla_{[p, q]}L^{t_k}(\tilde{p}^{t_k}, \tilde{q}^{t_k}, d^{t_k}) \\ \tilde{V}^{t_k}(\tilde{p}^{t_k}, \tilde{q}^{t_k}) - (V_{\max} - \epsilon_\mu^{t_k}) \\ (|\tilde{V}^{t_k}| - V_{\max}) \\ P^{t_k}_0(p^{t_k}, q^{t_k}) - P^{t_k}_0 \end{bmatrix},$$

From an optimization standpoint, the vector $e^{t_k}$ models errors in the computation of the gradients that are due to measurements of voltages and gradients at the substations. From Assumption 3 and Assumption 4, and using the result of Lemma 1, the norm of $e^{t_k}$ can be bounded as:

$$\|e^{t_k}\|_2 \leq (L + \nu)^2 e_p^2 + 2e_v^2 + 2e_0^2. \hspace{1cm} (48)$$

The proof now follows steps similar to [18]. Particularly, expand the right-hand side of (46) as

$$\|z^{t_k-1} - \alpha \Phi^{t_k-1}_e(z^{t_k-1}) - z^{*, t_k-1} + \alpha \Phi^{t_k-1}_e(z^{*, t_k-1})\|_2 \leq \|z^{t_k-1} - \alpha \Phi^{t_k-1}_e(z^{t_k-1}) - z^{*, t_k-1} + \alpha \Phi^{t_k-1}_e(z^{*, t_k-1})\|_2 + \|\alpha e^{t_k-1}\|_2. \hspace{1cm} (49)$$

Using the results of Lemma 2, we can write

$$\|z^{t_k-1} - \alpha \Phi^{t_k-1}_e(z^{t_k-1}) - z^{*, t_k-1} + \alpha \Phi^{t_k-1}_e(z^{*, t_k-1})\|_2 \leq (1 - 2\omega \eta + \alpha^2 \bar{B}^2)\|z^{t_k-1} - z^{*, t_k-1}\|_2. \hspace{1cm} (50)$$

and, by putting together the results in (46), (50), and (48), we have that

$$\|z^{t_k} - z^{*, t_k-1}\|_2 \leq \sqrt{(L + \nu)^2 e_p^2 + 2e_v^2 + 2e_0^2} + \sqrt{1 - 2\omega \eta + \alpha^2 \bar{B}^2} \leq \alpha + \epsilon + \sigma. \hspace{1cm} (51)$$

Let $\rho(\alpha) := \sqrt{1 - 2\omega \eta + \alpha^2 \bar{B}^2}$ and notice that $B$ in (18) is given by $B = \bar{B}^2$. Given a constant $\sigma \geq 0$ such that $|z^{*, t_k-1} - z^{*, t_k-1}| \leq \sigma$ for all $t_k \geq 0$, and by using the triangle inequality, it follows that

$$\|z^{t_k} - z^{*, t_k-1}\|_2 \leq \|z^{t_k} - z^{*, t_k-1} - z^{*, t_k-1} + z^{*, t_k-1}\|_2 \leq \|z^{t_k} - z^{*, t_k-1}\|_2 + \sigma \leq \rho(\alpha)\|z^{t_k-1} - z^{*, t_k-1}\|_2 + \alpha \epsilon + \sigma. \hspace{1cm} (52)$$

If $\rho(\alpha) < 1$, then (52) is a contraction and (18) readily follows.

Notice that an upper bound on the tracking error at each iteration can be obtained from (52).

**D. Application to multiphase systems**

Consider an approximate linear model

$$|v| \approx A|p| + B|q| + c, \hspace{1cm} (53)$$

$$\begin{bmatrix} P^0_0, Q^0_0, P^0_1, Q^0_1, P^0_2, Q^0_2 \end{bmatrix}^T \approx M|p| + N|q| + o, \hspace{1cm} (54)$$

where $v$ collects the voltages per phase and per node, $p$ and $q$ are vectors collecting the net injected active and reactive powers per phase and per node (with $A$, $B$, $M$, $N$, $a$, $o$ of appropriate dimensions), and where $(P^0_0, Q^0_0)$ denote the active and reactive powers flowing into the feeder on phase $\phi$. Model (53)–(54) can be obtained as shown in [22, 35] or by following steps similar to Appendix A. Suppose that a setpoint for for active power at the substation on each phase is given at time $t_k$, and denote the $3 \times 1$ vector collecting the setpoints as $P_{0, set}^{t_k} := \begin{bmatrix} P_{0, set}^{t_k, P_0}, P_{0, set}^{t_k, P_1}, P_{0, set}^{t_k, P_2} \end{bmatrix}$.

With $(P^0_0, Q^0_1, Q^0_2)$ denoting the setpoint for a DER at phase $\phi \subseteq \{a, b, c\}$ of node $i$, consider the implementation of algorithm (12) per phase and node shown next:

**[S1a]** Collect voltage-magnitude measurements $\{|V^i_n^{\phi, t_k}|\}_{n \in M}, \phi \subseteq \{a, b, c\}.$

**[S1b]** Collect measurement of $P^0_0, Q^0_1, Q^0_2, \phi \subseteq \{a, b, c\}.$
For every phase $\phi$ of node $n \in \mathcal{M}$, update $\gamma_{i,\triangle n,tk+1}$ and $\mu_{i,\triangle n,tk+1}$ as follows:

$$\gamma_{i,\triangle n,tk+1} = \text{proj}_{R_R} \left\{ \gamma_{i,\triangle n,tk} + \alpha \left( V_{\text{min}} - |\gamma_{i,\triangle n,tk} - E_{\gamma_{i,\triangle n,tk}}| \right) \right\}$$  \hfill (55a)

$$\mu_{i,\triangle n,tk+1} = \text{proj}_{R_R} \left\{ \mu_{i,\triangle n,tk} + \alpha \left( V_{\text{max}} - |\mu_{i,\triangle n,tk} - E_{\mu_{i,\triangle n,tk}}| \right) \right\}$$  \hfill (55b)

For the feeder head, if $h^k = 1$ update dual variables associated with each phase as:

$$\chi_{\phi,t,h^k+1} = \text{proj}_{R_R} \left\{ \chi_{\phi,t,h^k} + \alpha \left( \tilde{P}_{\phi,t,h^k} - P_{0,\phi,t,h^k} - E_{\phi,t,h^k} \right) \right\}$$  \hfill (55c)

$$\zeta_{\phi,t,h^k+1} = \text{proj}_{R_R} \left\{ \zeta_{\phi,t,h^k} + \alpha \left( P_{0,\phi,t,h^k} - \tilde{P}_{\phi,t,h^k} - E_{\zeta_{\phi,t,h^k}} \right) \right\}$$  \hfill (55d)

Measure output powers $\tilde{P}_{\phi,t,h^k}, \tilde{Q}_{\phi,t,h^k}$ at DER at phase $\phi$ of $i \in G$.

Update power setpoints for each DER as:

$$\left[ \begin{array}{c} P_{\phi,t,h^k+1}^i \\ Q_{\phi,t,h^k+1}^i \end{array} \right] = \text{proj}_{Q_i} \left[ \begin{array}{c} P_{\phi,t,h^k}^i \\ Q_{\phi,t,h^k}^i \end{array} \right] - \alpha \nabla_{P_{\phi,t},Q_{\phi,t}} \mathcal{L}_{\phi,t}(p, q, d) |P_{\phi,t,h^k}^i - Q_{\phi,t,h^k}^i, d,h^k+1|$$  \hfill (55e)

Command setpoints to each DER and return to [S1a].

REFERENCES


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