EXACT FINITE-DIMENSIONAL FILTERS FOR MAXIMUM LIKELIHOOD PARAMETER ESTIMATION OF CONTINUOUS-TIME LINEAR GAUSSIAN SYSTEMS

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Abstract. In this paper, we derive a new class of finite-dimensional filters for integrals and stochastic integrals of moments of the state for continuous-time linear Gaussian systems. Apart from being of significant mathematical interest, these new filters can be used with the expectation maximization (EM) algorithm to yield maximum likelihood estimates of the model parameters.

Key words. stochastic systems, finite-dimensional filters, Kalman filter, expectation maximization algorithm, parameter estimation

AMS subject classifications. 93E11, 93E12, 93E10, 60G35

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1. Introduction. The Kalman filter is widely used in engineering, economics, and other fields. It considers linear Gaussian dynamics for the state x and observation y processes of the form

$$\begin{aligned} (1.1) \quad dx_t &= A_t x_t dt + B_t dw_t, \quad x_0 \in \mathbb{R}^m, \\ (1.2) \quad dy_t &= C_t x_t dt + D_t dv_t, \quad y_0 = 0 \in \mathbb{R}^n. \end{aligned}$$

The Kalman filter determines the conditional density of the unobserved signal x_t given the observations to time t , $\mathcal{Y}_t = \sigma\{y_s : s \leq t\}$, as a Gaussian random variable with mean m_t and variance Σ_t . Here m_t is the conditional mean of x_t given $\mathcal{Y}_t$, and Σ_t is the conditional variance of x_t , although Σ_t turns out to be deterministic and given by the Riccati equation.

To apply the Kalman filter, however, the parameters of the model, that is, the entries in the matrices A , B , C , and D , need to be known. Maximum likelihood estimation of these parameters via the expectation maximization (EM) algorithm has been studied in discrete time in [1], [2], [3] and in continuous time in [9]. In continuous time, the EM algorithm requires computation of the filtered estimates of quantities such as $\int_0^t x_s dx_s^\theta$, $\int_0^t x_s dy_s^\theta$, $\int_0^t x_s x_s^\theta ds$.

In all the existing literature on parameter estimation of linear Gaussian models via the EM algorithm, filtered estimates of the above quantities are computed via Kalman smoothing, which requires large memory in any numerical implementation. The main result of this paper, Theorem 3.10, provides *finite-dimensional filters for (the components of) such integral processes*. In fact, as pointed out in section 4, finite-dimensional filters exist for integrals and stochastic integrals of moments of all orders

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of the state process. In [18] and [19], finite-dimensional filters are described using Lie algebra methods, for time integrals of powers of the state variable. However, our techniques are quite different; we also obtain results for stochastic integrals of powers of the process, and we apply our results to maximum likelihood parameter estimation. Few finite-dimensional filters are known, so our results were certainly a surprise.

The analogous results in discrete time are derived in a companion paper [17]. The continuous-time methods and results in this paper are different from, and not just adaptations of, the discrete-time results in [17]. As described in [17], estimation of the state-space parameters arises in several areas of signal processing and control, including multisensor signal enhancement and speech coding, and also in econometrics.

Parameter estimation for diffusion processes is nicely discussed in [9]; this work also includes a discussion of the EM algorithm. Section 5 describes this application of the results of section 4. The techniques of this paper extend those for Markov chains presented in [5] and the recent book, [6].

The paper is organized as follows: in section 2 we present a standard measure change that simplifies the derivation of our filters. In section 3 the new finite-dimensional filters are derived. Section 4 derives a finite-dimensional filter for higher order moments of the state. In section 5, the filters in section 3 are used to implement a filter-based EM algorithm for maximum likelihood parameter estimation.

2. Dynamics. Consider the classical linear Gaussian model for the signal model and observation processes. That is, the signal $\{x_t\}$, $t \geq 0$, is described by the equation

$$(2.1) \quad dx_t = A_t x_t dt + B_t dw_t, \quad x_0 \in \mathbb{R}^m,$$

and the observation process $\{y_t\}$, $t \geq 0$, is described as

$$(2.2) \quad dy_t = C_t x_t dt + D_t dv_t, \quad y_0 = 0 \in \mathbb{R}^n.$$

Here w and v are independent r -dimensional and n -dimensional Brownian motions, respectively, defined on a probability space $(\Omega, \mathcal{F}, P)$. Further, w and v are independent of x_0 . We assume that x_0 is a random variable with normal density $\pi_0(x)$, which is $N(\hat{x}_0, P_0)$.

The matrix functions $A_t \in \mathbb{R}^{m \times m}$, $B_t \in \mathbb{R}^{m \times r}$, $C_t \in \mathbb{R}^{n \times m}$, and $D_t \in \mathbb{R}^{n \times n}$ are measurable functions of t . We assume D_t is a positive definite matrix.

We model the above dynamics by supposing that initially we have a probability space $(\Omega, \mathcal{F}, \bar{P})$ such that under $\bar{P}$

1. w is r -dimensional Brownian motion and $\{x_t\}$ is defined by (2.1).
2. $\{y_t\}$ is n -dimensional Brownian motion, independent of w and x_0 , and having quadratic variation $\langle y \rangle_t = D_t > 0$; i.e., D_t is a positive definite matrix.

Notation. Consider the complete right continuous filtrations

$$\begin{aligned} \mathcal{G}_t &= \sigma\{x_s, y_s : s \leq t\}, \\ \mathcal{Y}_t &= \sigma\{y_s : s \leq t\}. \end{aligned}$$

Vectors x , y will be considered as column vectors. Write

$$(2.3) \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^\theta.$$

For any function $g : \mathbb{R}^m \rightarrow \mathbb{R}$, write

$$\nabla^2 g = \begin{bmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 g}{\partial x_m^2} \end{bmatrix}.$$

For a vector field $g(x) = [g_1(x) \ g_2(x) \ \cdots \ g_m(x)]^\theta$ defined on $\mathbb{R}^m$, define

$$\operatorname{div}(g) = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \cdots + \frac{\partial g_m}{\partial x_m}.$$

Consider the exponential

$$(2.4) \quad \Lambda_t = \exp \left(\int_0^t (C_s x_s)^\theta (D_s^{-1})^\theta D_s^{-1} dy_s - \frac{1}{2} \int_0^t x_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} C_s x_s ds \right).$$

Then

$$(2.5) \quad d\Lambda_t = \Lambda_t x_t^\theta C_t^\theta (D_s^{-1})^\theta D_s^{-1} dy_t$$

and $\bar{\mathbf{E}}\{\Lambda_t\} = 1$, where $\bar{\mathbf{E}}$ denotes expectation under $\bar{P}$.

If we define a measure P in terms of $\bar{P}$ by setting

$$\left. \frac{dP}{d\bar{P}} \right|_{G_t} = \Lambda_t,$$

then Girsanov's theorem implies that under P , v_t is a standard n -dimensional Brownian motion if we define

$$dv_t = D_t^{-1} (dy_t - C_t x_t dt), \quad v_0 = 0.$$

That is, under P ,

$$dy_t = C_t x_t dt + D_t dv_t.$$

Note that under P , the process $\{x_t\}$ still satisfies (2.1). Consequently, under P the processes $\{x_t\}$ and $\{y_t\}$ satisfy the real world dynamics (2.1) and (2.2). However, $\bar{P}$ is a more convenient measure with which to work.

Below, we assume that $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is an arbitrary "test" function, which is in $\mathbb{C}^2$ and has compact support.

Filtering is concerned with estimates of the form $\mathbf{E}\{g(x_t)|\mathcal{Y}_t\}$. Using a version of Bayes's theorem [6], we have

$$(2.6) \quad \mathbf{E}\{g(x_t)|\mathcal{Y}_t\} = \frac{\bar{\mathbf{E}}\{\Lambda_t g(x_t)|\mathcal{Y}_t\}}{\bar{\mathbf{E}}\{\Lambda_t|\mathcal{Y}_t\}}.$$

Write $\sigma(g)_t = \bar{\mathbf{E}}\{\Lambda_t g(x_t)|\mathcal{Y}_t\}$ so that

$$\mathbf{E}\{g(x_t)|\mathcal{Y}_t\} = \frac{\sigma(g)_t}{\sigma(1)_t}.$$

Consequently, $\sigma(g)_t$ is a measure-valued process; it is an unnormalized conditional expectation of $g(x_t)$ given $\mathcal{Y}_t$.

Suppose that $\sigma(\cdot)_t$ has a density $\alpha_t(x)$; then

$$\sigma(g)_t = \bar{\mathbf{E}}\{\Lambda_t g(x_t) | \mathcal{Y}_t\} = \int_{\mathbb{R}^m} g(x) \alpha_t(x) dx.$$

We now give a proof of the well-known Zakai equation for α .

THEOREM 2.1.

$$(2.7) \quad \begin{aligned} \alpha_t(x) = \alpha_0(x) - \int_0^t \operatorname{div}(A_s x \alpha_s(x)) ds + \frac{1}{2} \int_0^t \operatorname{Tr}(\nabla^2 \alpha_s(x) B_s B_s^\theta) ds \\ + \int_0^t \alpha_s(x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s, \end{aligned}$$

where $\alpha_0(x) = \pi_0(x)$, the density of x_0 .

Proof. From Ito's rule

$$(2.8) \quad \begin{aligned} g(x_t) = g(x_0) + \int_0^t (\nabla g(x_s))^\theta A_s x_s ds + \int_0^t (\nabla g(x_s))^\theta B_s dw_s \\ + \frac{1}{2} \int_0^t \operatorname{Tr}(\nabla^2 g(x_s) B_s B_s^\theta) ds. \end{aligned}$$

From (2.5) and (2.8)

$$(2.9) \quad \begin{aligned} \Lambda_t g(x_t) = g(x_0) + \int_0^t \Lambda_s (\nabla g(x_s))^\theta A_s x_s ds + \int_0^t \Lambda_s (\nabla g(x_s))^\theta B_s dw_s \\ + \frac{1}{2} \int_0^t \Lambda_s \operatorname{Tr}(\nabla^2 g(x_s) B_s B_s^\theta) ds + \int_0^t \Lambda_s g(x_s) x_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s. \end{aligned}$$

Conditioning each side of (2.9) on $\mathcal{Y}_t$ (see Lemma 3.2, p. 261, of [13]) we have

$$(2.10) \quad \begin{aligned} \sigma(g)_t = \sigma(g)_0 + \int_0^t \sigma((\nabla g)^\theta A x)_s ds + \frac{1}{2} \int_0^t \sigma(\operatorname{Tr}(\nabla^2 g) B_s B_s^\theta) ds \\ + \int_0^t \sigma(g x^\theta)_s C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s. \end{aligned}$$

Integrating each term by parts as on p. 277 of [7] gives (2.7). $\square$

Remarks. The Zakai equation (2.7) is a stochastic partial differential equation for the unnormalized conditional density of x_t given $\mathcal{Y}_t$. In general, the solution of this equation is a conditional density function, evolving stochastically in time. For the linear, Gaussian dynamics (2.1) and (2.2), however, $\alpha_t(x)$ has a simple form. In fact (see [10] or [11]),

$$(2.11) \quad \alpha_t(x) = \frac{\nu_t}{(2\pi)^{m/2} |\Sigma_t|^{1/2}} \exp\left(-\frac{1}{2}(x - m_t)^\theta \Sigma_t^{-1} (x - m_t)\right).$$

Here $m_t = \mathbf{E}\{x_t | \mathcal{Y}_t\}$, $m_0 = \hat{x}_0$, $\Sigma_t = \mathbf{E}\{(x_t - m_t)(x_t - m_t)^\theta | \mathcal{Y}_t\}$, $\Sigma_0 = P_0$, and ν_t is a normalizing factor.

It is well known that m_t and Σ_t are given by the Kalman filter equations

$$(2.12) \quad dm_t = A_t m_t dt + \Sigma_t C_t^\theta (D_t^{-1})^\theta D_t^{-1} (dy_t - C_t m_t dt),$$

$$(2.13) \quad \dot{\Sigma}_t = \Sigma_t A_t^\theta + A_t \Sigma_t + B_t B_t^\theta - \Sigma_t C_t^\theta (D_t^{-1})^\theta D_t^{-1} C_t \Sigma_t.$$

Note that Σ_t is deterministic and can be computed off-line. Also,

$$(2.14) \quad \nu_t = (2\pi)^{m/2} \exp \left(\int_0^t m_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s - \frac{1}{2} \int_0^t m_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} C_s m_s ds \right).$$

3. Finite-dimensional filters. Let $e_i, e_j \in \mathbb{R}^m$ denote unit vectors with 1 in the i th and j th positions, respectively. Write

$$(3.1) \quad H_t^{ij} = \int_0^t \langle x_s, e_i \rangle \langle e_j, x_s \rangle ds = \int_0^t x_s^\theta (e_i e_j^\theta) x_s ds,$$

$$(3.2) \quad L_t^{ij} = \int_0^t \langle x_s, e_i \rangle \langle e_j, dx_s \rangle = \int_0^t x_s^\theta (e_i e_j^\theta) dx_s;$$

here $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Also let $f_j \in \mathbb{R}^n$ denote the unit vector with 1 in the j th position. Write

$$(3.3) \quad J_t^{ij} = \int_0^t \langle x_s, e_i \rangle \langle f_j, dy_s \rangle = \int_0^t x_s^\theta (e_i f_j^\theta) dy_s.$$

The parameters of our model are the entries in the matrices A_t , B_t , C_t , and D_t . To estimate these parameters using the EM algorithm (see section 5), it is necessary to obtain filtered estimates of these processes. That is, we wish to obtain expressions for

$$\mathbf{E}\{H_t^{ij}|\mathcal{Y}_t\}, \quad \mathbf{E}\{L_t^{ij}|\mathcal{Y}_t\}, \quad \mathbf{E}\{J_t^{ij}|\mathcal{Y}_t\}.$$

Previously, these estimates have been obtained by smoothing procedures. For example,

$$\mathbf{E}\{H_t^{ij}|\mathcal{Y}_t\} = \int_0^t \mathbf{E}\{x_s^\theta (e_i e_j^\theta) x_s|\mathcal{Y}_t\} ds.$$

However, this involves a considerable memory requirement. In this section we prove the remarkable result that the filtered estimates for H_t^{ij} , L_t^{ij} , and J_t^{ij} can be described in terms of a finite number of statistics.

Motivated by the techniques in [6], we define a measure associated with H_t^{ij} as follows.

DEFINITION 3.1. *For any test function $g : \mathbb{R}^m \rightarrow \mathbb{R}$, define a measure-valued process $\bar{\mathbf{E}}\{\Lambda_t H_t^{ij} g(x_t)|\mathcal{Y}_t\}$. This has a density $\beta_t^{ij}(x)$ so that*

$$\bar{\mathbf{E}}\{\Lambda_t H_t^{ij} g(x_t)|\mathcal{Y}_t\} = \int_{\mathbb{R}^m} \beta_t^{ij}(x) g(x) dx.$$

The existence of the density $\beta_t^{ij}(x)$ follows from the existence and uniqueness of solutions of stochastic partial differential equations such as (3.9). This is established in section 4.2 of [10] and on page 140 of [12].

The following theorem shows the surprising result that we can describe the measure $\beta_t^{ij}(x)$ exactly as a quadratic in x multiplying the $\alpha_t(x)$ of (2.7).

THEOREM 3.2. *At time t , the density $\beta_t^{ij}(x)$ is completely described by the five statistics a_t^{ij} , b_t^{ij} , c_t^{ij} , Σ_t , and m_t as follows:*

$$(3.4) \quad \beta_t^{ij}(x) = \left(a_t^{ij} + x^\theta b_t^{ij} + x^\theta c_t^{ij} x \right) \alpha_t(x).$$

Here $a_t^{ij} \in \mathbb{R}$, $b_t^{ij} \in \mathbb{R}^m$, and $c_t^{ij} \in L_s(\mathbb{R}^m, \mathbb{R})$, the space of symmetric $m \times m$ matrices. Further,

$$(3.5) \quad \frac{da_t^{ij}}{dt} = \text{Tr} \left(c_t^{ij} B_t B_t^\theta \right) + b_t^{ij\theta} B_t B_t^\theta \Sigma_t^{-1} m_t, \quad a_0^{ij} = 0 \in \mathbb{R},$$

$$(3.6) \quad \frac{db_t^{ij}}{dt} = - \left(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta \right) b_t^{ij} + 2 c_t^{ij} B_t B_t^\theta \Sigma_t^{-1} m_t, \quad b_0^{ij} = 0 \in \mathbb{R}^m,$$

$$(3.7) \quad \begin{aligned} \frac{dc_t^{ij}}{dt} &= - \left(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta \right) c_t^{ij} - c_t^{ij} \left(A_t + B_t B_t^\theta \Sigma_t^{-1} \right) + \frac{1}{2} (e_j e_i^\theta + e_i e_j^\theta), \\ c_0^{ij} &= 0 \in L_s(\mathbb{R}^m, \mathbb{R}). \end{aligned}$$

Proof. First note that for any test function g , applying Ito's rule to (2.1) and (2.5),

$$(3.8) \quad \begin{aligned} \Lambda_t H_t^{ij} g(x_t) &= \int_0^t \Lambda_s H_s^{ij} (\nabla g(x_s))^\theta A_s x_s ds + \int_0^t \Lambda_s H_s^{ij} (\nabla g(x_s))^\theta B_s dw_s \\ &\quad + \frac{1}{2} \int_0^t \Lambda_s H_s^{ij} \text{Tr} (\nabla^2 g(x_s) B_s B_s^\theta) ds \\ &\quad + \int_0^t \Lambda_s H_s^{ij} g(x_s) x_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s \\ &\quad + \int_0^t \Lambda_s g(x_s) x_s^\theta (e_i e_j^\theta) x_s ds. \end{aligned}$$

Conditioning on $\mathcal{Y}_t$ under $\bar{P}$ (see Lemma 3.2, p. 261, of [13]) we have

$$\begin{aligned} \bar{\mathbf{E}}\{\Lambda_t H_t^{ij} g(x_t) | \mathcal{Y}_t\} &= \int_0^t \bar{\mathbf{E}}\{\Lambda_s H_s^{ij} (\nabla g(x_s))^\theta A_s x_s | \mathcal{Y}_s\} ds \\ &\quad + \frac{1}{2} \int_0^t \bar{\mathbf{E}}\{\Lambda_s H_s^{ij} \text{Tr} (\nabla^2 g(x_s) B_s B_s^\theta) | \mathcal{Y}_s\} ds \\ &\quad + \int_0^t \bar{\mathbf{E}}\{\Lambda_s H_s^{ij} g(x_s) x_s^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} | \mathcal{Y}_s\} dy_s \\ &\quad + \int_0^t \bar{\mathbf{E}}\{\Lambda_s g(x_s) x_s^\theta (e_i e_j^\theta) x_s | \mathcal{Y}_s\} ds. \end{aligned}$$

That is, in terms of the densities $\beta_t^{ij}(x)$ and $\alpha_t(x)$,

$$\begin{aligned} \int_{\mathbb{R}^m} \beta_t^{ij}(x) g(x) dx &= \int_0^t \int_{\mathbb{R}^m} \beta_s^{ij}(x) (\nabla g(x))^\theta A_s x dx ds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^m} \beta_s^{ij}(x) \text{Tr} (\nabla^2 g(x) B_s B_s^\theta) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^m} \beta_s^{ij}(x) g(x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dx dy_s \\ &\quad + \int_0^t \int_{\mathbb{R}^m} \alpha_s(x) g(x) x^\theta (e_i e_j^\theta) x dx ds. \end{aligned}$$

Integrating by parts in x , because this equation holds for all test functions g , we see

that $\beta_t^{ij}(x)$ must satisfy the stochastic partial differential equation:

$$(3.9) \quad \begin{aligned} \beta_t^{ij}(x) = & - \int_0^t \operatorname{div} (\beta_s^{ij}(x) A_s x) ds + \frac{1}{2} \int_0^t \operatorname{Tr} (\nabla^2 \beta_s^{ij}(x) B_s B_s^\theta) ds \\ & + \int_0^t \beta_s^{ij}(x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s + \int_0^t \alpha_s(x) x^\theta (e_i e_j^\theta) x ds. \end{aligned}$$

We look for a solution of (3.9) of the form

$$(3.10) \quad \bar{\beta}_s(x) = (a_s^{ij} + x^\theta b_s^{ij} + x^\theta c_s^{ij} x) \alpha_s(x).$$

As noted just after Definition 3.1, if such a solution exists, it is unique.

To simplify notation we drop the superscripts i, j on a, b , and c . Then

$$\begin{aligned} \operatorname{div} (\bar{\beta}_s(x) A_s x) &= \operatorname{div} ((a_s + b_s^\theta x + x^\theta c_s x) \alpha_s(x) A_s x) \\ &= (b_s + 2 c_s x)^\theta A_s x \alpha_s(x) + (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{div} (\alpha_s(x) A_s x), \\ \nabla \bar{\beta}_s(x) &= \nabla ((a_s + b_s^\theta x + x^\theta c_s x) \alpha_s(x)) \\ &= (b_s + 2 c_s x) \alpha_s(x) + (a_s + b_s^\theta x + x^\theta c_s x) \nabla \alpha_s(x), \\ \nabla^2 \bar{\beta}_s(x) &= 2 c_s \alpha_s(x) + 2 (b_s + 2 c_s x) (\nabla \alpha_s(x))^\theta \\ &\quad + (a_s + b_s^\theta x + x^\theta c_s x) \nabla^2 \alpha_s(x), \\ \operatorname{Tr} (\nabla^2 \bar{\beta}_s(x) B_s B_s^\theta) &= 2 \alpha_s(x) \operatorname{Tr} (c_s B_s B_s^\theta) + 2 (b_s + 2 c_s x)^\theta B_s B_s^\theta \nabla \alpha_s(x) \\ &\quad + (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{Tr} (\nabla^2 \alpha_s(x) B_s B_s^\theta). \end{aligned}$$

Now from (2.11)

$$\nabla \alpha_s(x) = -\Sigma_s^{-1}(x - m_s) \alpha_s(x).$$

Consequently, if we substitute $\bar{\beta}_t(x)$, given by (3.10) in the differential form of the right-hand side of (3.9), we obtain

$$(3.11) \quad \begin{aligned} & - (b_s + 2 c_s x)^\theta A_s x \alpha_s(x) ds - (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{div} (\alpha_s(x) A_s x) ds \\ & + \alpha_s(x) \operatorname{Tr} (c_s B_s B_s^\theta) ds - (b_s + 2 c_s x)^\theta B_s B_s^\theta \Sigma_s^{-1}(x - m_s) \alpha_s(x) ds \\ & + \frac{1}{2} (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{Tr} (\nabla^2 \alpha_s(x) B_s^\theta B_s) ds \\ & + (a_s + b_s^\theta x + x^\theta c_s x) \alpha_s(x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s \\ & + \alpha_s(x) x^\theta (e_i e_j^\theta) x ds. \end{aligned}$$

Also,

$$(3.12) \quad d\bar{\beta}_s(x) = (da_s + db_s^\theta x + x^\theta dc_s x) \alpha_s(x) + (a_s + b_s^\theta x + x^\theta c_s x) d\alpha_s(x).$$

Consequently, $\bar{\beta}_s(x)$, given by (3.10), is a solution of (3.9) if (3.11) equals (3.12). However, $\alpha_s(x)$ solves the Zakai equation (2.7), so

$$d\alpha_s(x) = -\operatorname{div} (\alpha_s(x) A_s x) ds + \frac{1}{2} \operatorname{Tr} (\nabla^2 \alpha_s(x) B_s B_s^\theta) ds + \alpha_s(x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} dy_s.$$

Therefore, substituting the above expression for $d\alpha_s(x)$ into (3.12) yields

$$(3.13) \quad \begin{aligned} d\bar{\beta}_s(x) &= (da_s + db_s^\theta x + x^\theta dc_s x) \alpha_s(x) \\ &\quad - (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{div} (\alpha_s(x) A_s x) ds \\ &\quad + \frac{1}{2} (a_s + b_s^\theta x + x^\theta c_s x) \operatorname{Tr} (\nabla^2 \alpha_s(x) B_s B_s^\theta) ds \\ &\quad + (a_s + b_s^\theta x + x^\theta c_s x) x^\theta C_s^\theta (D_s^{-1})^\theta D_s^{-1} \alpha_s(x) dy_s. \end{aligned}$$

Finally equating the coefficients of x , x^θ , x and the constants in (3.11) and (3.13), we see that the result holds if (3.5), (3.6), and (3.7) hold. $\square$

We now explicitly solve the ordinary differential equations (3.6) and (3.7).

Write G_t for the matrix solution of

$$(3.14) \quad \frac{dG_t}{dt} = -(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta) G_t, \quad G_0 = I_{m-m}.$$

Note that G_t is deterministic and can be calculated off-line. Also, as an exponential matrix, G_t has an inverse G_t^{-1} .

LEMMA 3.3. *The explicit solutions of (3.6) and (3.7) are*

$$(3.15) \quad b_t^{ij} = 2 G_t \left(\int_0^t G_s^{-1} c_s^{ij} B_s B_s^\theta \Sigma_s^{-1} m_s ds \right),$$

$$(3.16) \quad c_t^{ij} = \frac{1}{2} G_t \left(\int_0^t G_s^{-1} (e_j e_i^\theta + e_i e_j^\theta) (G_s^\theta)^{-1} ds \right) G_t^\theta.$$

Proof. The above equations follow using variation of constants. $\square$

Remark. We proceed similarly with the process J_t^{ij} and L_t^{ij} , omitting details.

DEFINITION 3.4. *For any test function $g \in \mathbb{C}_0^2(\mathbb{R}^m)$ define the measure-valued process $\bar{\mathbf{E}}\{\Lambda_t J_t^{ij} g(x_t) | \mathcal{Y}_t\}$. From the results of [10] and [12], this has a density $\gamma_t^{ij}(x)$ so that*

$$\bar{\mathbf{E}}\{\Lambda_t J_t^{ij} g(x_t) | \mathcal{Y}_t\} = \int_{\mathbb{R}^m} \gamma_t^{ij}(x) g(x) dx.$$

THEOREM 3.5. *At time t , the density $\gamma_t^{ij}(x)$ is completely described by the five statistics $\bar{a}_t^{ij}$, $\bar{b}_t^{ij}$, $\bar{c}_t^{ij}$, Σ_t , and m_t as follows:*

$$(3.17) \quad \gamma_t^{ij}(x) = \left(\bar{a}_t^{ij} + x^\theta \bar{b}_t^{ij} + x^\theta \bar{c}_t^{ij} x \right) \alpha_t(x).$$

Here, $\bar{a}_t^{ij} \in \mathbb{R}$, $\bar{b}_t^{ij} \in \mathbb{R}^m$, and $\bar{c}_t^{ij} \in L_s(\mathbb{R}^m, \mathbb{R}^m)$. Further,

$$(3.18) \quad \frac{d\bar{a}_t^{ij}}{dt} = \text{Tr} \left(\bar{c}_t^{ij} B_t B_t^\theta \right) + \bar{b}_t^{ij\theta} B_t B_t^\theta \Sigma_t^{-1} m_t, \quad \bar{a}_0^{ij} = 0 \in \mathbb{R},$$

$$(3.19) \quad d\bar{b}_t^{ij} = \left[-(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta) \bar{b}_t^{ij} + 2 \bar{c}_t^{ij} B_t B_t^\theta \Sigma_t^{-1} m_t \right] dt + dy_t^\theta f_j e_i, \quad \bar{b}_0^{ij} = 0 \in \mathbb{R}^m,$$

$$(3.20) \quad \frac{d\bar{c}_t^{ij}}{dt} = - (A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta) \bar{c}_t^{ij} - \bar{c}_t^{ij} (A_t + B_t B_t^\theta \Sigma_t^{-1}) + \frac{1}{2} (e_i f_j^\theta C_t + C_t^\theta f_j e_i^\theta),$$

$$\bar{c}_0^{ij} = 0 \in L_s(\mathbb{R}^m, \mathbb{R}^m).$$

Proof. The product $\Lambda_t J_t^{ij} g(x_t)$ is calculated and each side conditioned on $\mathcal{Y}_t$. After integration by parts, the following stochastic partial differential equation is obtained for $\gamma_t^{ij}(x)$:

$$(3.21) \quad d\gamma_t^{ij}(x) = -\text{div} \left(\gamma_t^{ij}(x) A_t x \right) dt + \frac{1}{2} \text{Tr} \left(\nabla^2 \gamma_t^{ij}(x) B_t B_t^\theta \right) dt + \gamma_t^{ij} x^\theta C_t^\theta (D_t^\theta)^{-1} D_t^{-1} dy_t$$

$$+ \alpha_t(x) x^\theta e_i f_j^\theta dy_t + \alpha_t(x) (x^\theta C_t^\theta f_j e_i^\theta x) dt$$

Recalling that $\alpha_t(x)$ satisfies the Zakai equation (2.7), we see that

$$\left(\bar{a}_t^{ij} + x^\theta \bar{b}_t^{ij} x + x^\theta \bar{c}_t^{ij} x \right) \alpha_t(x)$$

is a solution of (3.21) if $\bar{a}_t^{ij}$, $\bar{b}_t^{ij}$, and $\bar{c}_t^{ij}$ satisfy (3.18), (3.19), and (3.20), respectively. $\square$

We now obtain explicit solutions to the above equations. Note that $f_j^\theta dy_t = dy_t^\theta f_j = dy_t^j$, where y_t^j denotes the j th component of y_t .

LEMMA 3.6. *The explicit solutions of (3.19) and (3.20) are*

$$(3.22) \quad \bar{b}_t^{ij} = 2 G_t \left(\int_0^t G_s^{-1} \bar{c}_s^{ij} B_s B_s^\theta \Sigma_s^{-1} m_s ds + \int_0^t G_s^{-1} e_i dy_s^j \right)$$

and

$$(3.23) \quad \bar{c}_t^{ij} = \frac{1}{2} G_t \left(\int_0^t G_s^{-1} (e_i f_j^\theta C_s + C_s^\theta f_j e_i^\theta) (G_s^{-1})^\theta ds \right) G_t^\theta.$$

DEFINITION 3.7. *For any test function $g \in \mathbb{C}_0^2(\mathbb{R}^m)$ define the measure-valued process $\bar{\mathbf{E}}\{\Lambda_t L_t^{ij} g(x_t) | \mathcal{Y}_t\}$. From the results of [10] and [12], this has a density $\lambda_t^{ij}(x)$ so that*

$$\bar{\mathbf{E}}\{\Lambda_t L_t^{ij} g(x_t) | \mathcal{Y}_t\} = \int_{\mathbb{R}^m} \lambda_t^{ij}(x) g(x) dx.$$

THEOREM 3.8. *At time t , the density $\lambda_t^{ij}(x)$ is completely characterized by the five statistics r_t^{ij} , s_t^{ij} , and u_t^{ij} as follows:*

$$(3.24) \quad \lambda_t^{ij}(x) = \left(r_t^{ij} + x^\theta s_t^{ij} x + x^\theta u_t^{ij} x \right) \alpha_t(x).$$

Here, $r_t^{ij} \in \mathbb{R}$, $s_t^{ij} \in \mathbb{R}^m$, and $u_t^{ij} \in L_s(\mathbb{R}^n, \mathbb{R}^m)$. Further,

$$(3.25) \quad \frac{dr_t^{ij}}{dt} = \text{Tr} \left(u_t^{ij} B_t B_t^\theta \right) + s_t^{ij\theta} B_t B_t^\theta \Sigma_t^{-1} m_t - \text{Tr} \left(B_t B_t^\theta e_i e_j^\theta \right), \quad r_0^{ij} = 0 \in \mathbb{R},$$

$$(3.26) \quad \begin{aligned} \frac{ds_t^{ij}}{dt} = & - \left(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta \right) s_t^{ij} + 2 u_t^{ij} B_t B_t^\theta \Sigma_t^{-1} m_t \\ & - (e_j e_i^\theta) B_t B_t^\theta \Sigma_t^{-1} m_t, \quad s_0^{ij} = 0 \in \mathbb{R}^m, \end{aligned}$$

$$(3.27) \quad \begin{aligned} \frac{du_t^{ij}}{dt} = & - \left(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta \right) u_t^{ij} - u_t^{ij} \left(A_t + B_t B_t^\theta \Sigma_t^{-1} \right) \\ & + \frac{1}{2} \left(e_i e_j^\theta \left(A_t + B_t B_t^\theta \Sigma_t^{-1} \right) + \left(A_t^\theta + \Sigma_t^{-1} B_t B_t^\theta \right) e_j e_i^\theta \right), \\ & u_0^{ij} = 0 \in L_s(\mathbb{R}^m, \mathbb{R}^m). \end{aligned}$$

Proof. The product $\Lambda_t L_t^{ij} g(x_t)$ is calculated and each side conditioned on $\mathcal{Y}_t$. After integration by parts, the following stochastic partial differential equation is obtained for $\lambda_t^{ij}(x)$:

$$(3.28) \quad \begin{aligned} d\lambda_t^{ij}(x) = & -\text{div} \left(\lambda_t^{ij}(x) A_t x \right) dt \\ & + \frac{1}{2} \text{Tr} \left(\nabla^2 \lambda_t^{ij}(x) B_t B_t^\theta \right) dt + \lambda_t^{ij}(x) x^\theta C_t^\theta \left(D_t^{-1} \right)^\theta D_t^{-1} dy_t \\ & + \alpha_t(x) x^\theta e_i e_j^\theta A_t x dt - \text{div} \left(x^\theta e_i e_j^\theta B_t B_t^\theta \alpha_t(x) \right) dt. \end{aligned}$$

We see that

$$\left(r_t^{ij} + x^\theta s_t^{ij} x + x^\theta u_t^{ij} x \right) \alpha_t(x)$$

solves (3.28) if r_t^{ij} , s_t^{ij} , and u_t^{ij} satisfy (3.25), (3.26), and (3.27). $\square$

Again, (3.26) and (3.27) can be solved by variation of constants. We summarize this in the following lemma.

LEMMA 3.9. *The explicit solutions of (3.26) and (3.27) are*

$$(3.29) \quad s_t^{ij} = G_t \left(\int_0^t G_s^{-1} (2 u_s^{ij} - e_j e_i^\theta) \Sigma_s^{-1} B_s B_s^\theta m_s ds \right),$$

$$(3.30) \quad u_t^{ij} = \frac{1}{2} G_t \left(\int_0^t [G_s^{-1} (e_i e_j^\theta (A_s + B_s B_s^\theta \Sigma_s^{-1}) + (A_s^\theta + \Sigma_s^{-1} B_s B_s^\theta) e_j e_i^\theta) (G_s^{-1})^\theta] ds \right) G_t.$$

Remark. We observe from the definition of G_t , (3.14), that the integrand in (3.30) includes only half of the four terms in the derivative of $G_t^{-1} (e_i e_j^\theta + e_j e_i^\theta) (G_t^{-1})^\theta$, and so the integral cannot be evaluated in closed form.

THEOREM 3.10. *Finite-dimensional filters for H_t^{ij} , J_t^{ij} , and L_t^{ij} defined in (3.1), (3.3), and (3.2) are given by*

$$(3.31) \quad \mathbf{E}\{H_t^{ij} | \mathcal{Y}_t\} = a_t^{ij} + m_t^\theta b_t^{ij} + \sum_{p=1}^m \sum_{q=1}^m c_t^{ij}(p, q) \Sigma_t(p, q) + m_t^\theta c_t^{ij} m_t,$$

$$(3.32) \quad \mathbf{E}\{J_t^{ij} | \mathcal{Y}_t\} = \bar{a}_t^{ij} + m_t^\theta \bar{b}_t^{ij} + \sum_{p=1}^m \sum_{q=1}^m \bar{c}_t^{ij}(p, q) \Sigma_t(p, q) + m_t^\theta \bar{c}_t^{ij} m_t,$$

$$(3.33) \quad \mathbf{E}\{L_t^{ij} | \mathcal{Y}_t\} = r_t^{ij} + m_t^\theta s_t^{ij} + \sum_{p=1}^m \sum_{q=1}^m u_t^{ij}(p, q) \Sigma_t(p, q) + m_t^\theta u_t^{ij} m_t.$$

Proof. Recall from (2.11) that α_t is an unnormalized Gaussian density with mean m_t and variance Σ_t . Therefore,

$$\int_{\mathbb{R}^m} \alpha_t(x) dx = \nu_t.$$

Note that for any $k \in \mathbb{R}^m$

$$\int_{\mathbb{R}^m} k^\theta x \alpha_t(x) dx = (k^\theta m_t) \nu_t.$$

Also for any matrix $M \in L(\mathbb{R}^m, \mathbb{R}^m)$ with entries $M(p, q)$, $1 \leq p, q \leq m$,

$$\begin{aligned} \int_{\mathbb{R}^m} x^\theta M x \alpha_t(x) dx &= \int_{\mathbb{R}^m} (x - m_t)^\theta M (x - m_t) \alpha_t(x) dx + m_t^\theta M m_t \int_{\mathbb{R}^m} \alpha_t(x) dx \\ &= \left(\sum_{p=1}^m \sum_{q=1}^m M(p, q) \Sigma_t(p, q) + m_t^\theta M m_t \right) \nu_t. \end{aligned}$$

Now from Bayes's theorem (2.6), we have

$$\begin{aligned} \mathbf{E}\{H_t^{ij}|\mathcal{Y}_t\} &= \frac{\bar{\mathbf{E}}\{\Lambda_t H_t^{ij}|\mathcal{Y}_t\}}{\bar{\mathbf{E}}\{\Lambda_t|\mathcal{Y}_t\}} \\ &= \frac{\int_{\mathbb{R}^m} \beta_t^{ij}(x) dx}{\int_{\mathbb{R}^m} \alpha_t(x) dx} \\ &= a_t^{ij} + m_t^\emptyset b_t^{ij} + \sum_{p=1}^m \sum_{q=1}^m c_t^{ij}(p, q) \Sigma_t(p, q) + m_t^\emptyset c_t^{ij} m_t \end{aligned}$$

by (3.4) and because the factors ν_t cancel. The proofs of equations (3.32) and (3.33) are similar. $\square$

4. Finite-dimensional filter for higher order moments. The techniques of the previous section can be generalized to show that integrals and stochastic integrals of higher moments of the state variables have filtered estimates which can be expressed in terms of a finite number of statistics. The results also hold for other functions of the state and will be investigated in a subsequent paper.

Assumption 4.1. For notational simplicity, in this section we assume that the state and observation processes are scalar valued, i.e., that $m = n = 1$ in (2.1) and (2.2).

Let Γ_t be the process defined as

$$\Gamma_t = \int_0^t x_s^p ds, \quad p \in \mathbb{Z}^+.$$

Suppose $g : \mathbb{R} \rightarrow \mathbb{R}$ is any test function and suppose for a density-valued process $\mu_t(x)$,

$$\bar{\mathbf{E}}\{\Lambda_t \Gamma_t g(x_t)|\mathcal{Y}_t\} = \int_{\mathbb{R}} \mu_t(x) g(x) dx.$$

We now show that $\mathbf{E}\{\Gamma_t|\mathcal{Y}_t\}$ can be computed via a finite-dimensional filter.

THEOREM 4.2. *At time t , the density μ_t is completely characterized by the $p+3$ statistics, $a_t(0), a_t(1), \dots, a_t(p)$, Σ_t , and m_t as follows:*

$$(4.1) \quad \mu_t(x) = \left[\sum_{i=0}^p a_t(i) x^i \right] \alpha_t(x),$$

where $a_0(i) = 0$, $i = 1, \dots, p$, and

$$\begin{aligned} \frac{da_t(p)}{dt} &= -p (A_t + \Sigma_t^{-1} B_t^2) a_t(p) + 1, \\ \frac{da_t(p-1)}{dt} &= -(p-1) (A_t + \Sigma_t^{-1} B_t^2) a_t(p-1) + p a_t(p) \Sigma_t^{-1} B_t^2 m_t, \\ \frac{da_t(i)}{dt} &= -i (A_t + \Sigma_t^{-1} B_t^2) a_t(i) + \frac{1}{2} (i+1) (i+2) a_t(i+2) \\ &\quad + (i+1) a_t(i+1) \Sigma_t^{-1} B_t^2, \\ &\quad i = 1, \dots, p-2, \\ (4.2) \quad \frac{da_t(0)}{dt} &= B_t^2 a_t(2) + \Sigma_t^{-1} B_t^2 a_t(1) m_t. \end{aligned}$$

Proof.

$$\begin{aligned}\Lambda_t \Gamma_t g(x_t) &= \int_0^t \Lambda_s \Gamma_s \nabla g(x_s) A_s x_s ds + \int_0^t \Lambda_s \Gamma_s \nabla g(x_s) B_s dw_s \\ &\quad + \frac{1}{2} \int_0^t \Lambda_s \Gamma_s \nabla^2 g(x_s) B_s^2 ds \\ &\quad + \int_0^t \Lambda_s \Gamma_s g(x_s) x_s C_s D_s^{-2} dy_s + \int_0^t \Lambda_s g(x_s) x_s^p ds.\end{aligned}$$

Conditioning on $\mathcal{Y}_t$, as in [13],

$$\begin{aligned}\int_{\mathbb{R}} \mu_t(x) g(x) dx &= \int_0^t \int_{\mathbb{R}} \mu_s(x) \nabla g(x) A_s x dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \mu_s(x) \nabla^2 g(x) B_s^2 dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \mu_s(x) g(x) x C_s D_s^{-2} dx dy_s + \int_0^t \int_{\mathbb{R}} x^p g(x) \alpha_s(x) dx ds.\end{aligned}$$

Integrating by parts in x , we see that $\mu_t(\cdot)$ satisfies the stochastic partial differential equation

$$\begin{aligned}\mu_t(x) &= - \int_0^t \frac{d}{dx} (\mu_s(x) A_s x) ds + \frac{1}{2} \int_0^t \frac{d^2}{dx^2} \mu_s(x) B_s^2 ds + \int_0^t \mu_s(x) x C_s D_s^{-2} dy_s \\ &\quad + \int_0^t x^p \alpha_s(x) ds.\end{aligned}$$

It can then be verified that (4.1) is a solution to the above equation if the time-varying coefficients $a_t(0), \dots, a_t(p)$ satisfy the ordinary differential equations (4.2). $\square$

Remark. The ordinary differential equations (4.2) can be solved explicitly by variation of constants.

Finally, we note that a similar derivation to that given in this section yields finite-dimensional filters for $\int_0^t x_s^p dx_s$ and $\int_0^t x_s^p dy_s$, $p = 1, 2, 3, \dots$

5. Filtered EM algorithm for Gaussian state-space models. The aim of this section is to derive a filter-based EM algorithm for computing maximum-likelihood (ML) parameter estimates of a linear Gaussian state-space system. The finite-dimensional filters of section 3 are used in implementing the E-step of the EM algorithm, resulting in a filter-based EM algorithm.

Consider the time-invariant version of the state-space model (2.1), (2.2):

$$(5.1) \quad dx_t = A x_t dt + B dw_t,$$

$$(5.2) \quad dy_t = C x_t dt + D dv_t.$$

Our aim is to compute ML estimates of the parameters $\theta = (A, C)$ given the observations $\mathcal{Y}_t = \sigma\{y_s : s \leq t\}$ and assuming B, D are known. We do this via the EM algorithm.

Remark. Unlike the discrete-time case, in continuous time, it is not possible to obtain ML estimates of the variance terms B and D because measures corresponding to Wiener processes with different variances are not absolutely continuous (see Chapter 6.1 in [13]). At the end of this section, we give estimates for B and D in terms of the quadratic variations of the state and observation processes.

EM ALGORITHM. Suppose that we have observation history $\mathcal{Y}_t$ available. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability measures on $(\Omega, \mathcal{F})$, all absolutely continuous with respect to a fixed probability measure P_0 . The log likelihood function for computing an estimate of the parameter θ based on the information available in $\mathcal{Y}_t$ is

$$\mathcal{L}(\theta) = \mathbf{E}_0 \left\{ \log \frac{dP_\theta}{dP_0} \mid \mathcal{Y}_t \right\},$$

and the maximum likelihood estimate (MLE) is defined by

$$\hat{\theta} \in \operatorname{argmax}_{\theta \in \Theta} \mathcal{L}(\theta).$$

The EM algorithm is an iterative numerical method for computing the MLE. Let $\hat{\theta}_0$ be the initial parameter estimate. The EM algorithm generates a sequence of parameter estimates as follows.

Each iteration of the EM algorithm consists of two steps:

Step 1 (E-step). Set $\tilde{\theta} = \hat{\theta}_j$ and compute $Q(\cdot, \tilde{\theta})$, where

$$Q(\theta, \tilde{\theta}) = \mathbf{E}_{\tilde{\theta}} \left\{ \log \frac{dP_\theta}{dP_{\tilde{\theta}}} \mid \mathcal{Y}_t \right\}.$$

Step 2 (M-step). Find $\hat{\theta}_{j+1} \in \operatorname{argmax}_{\theta \in \Theta} Q(\theta, \tilde{\theta}_j)$.

The sequence-generated $\{\hat{\theta}_j, j \geq 0\}$ gives nondecreasing values of $\mathcal{L}(\hat{\theta}_j)$ with equality if and only if $\hat{\theta}_{j+1} = \hat{\theta}_j$.

It is shown in the appendix that

(5.3)

$$\begin{aligned} Q(\theta, \tilde{\theta}) = & \mathbf{E} \left\{ \int_0^t x_s^\theta A^\theta [B B^\theta]^\# dx_s - \frac{1}{2} \int_0^t x_s^\theta A^\theta [B B^\theta]^\# A x_s ds \mid \mathcal{Y}_t \right\} \\ & + \mathbf{E} \left\{ \int_0^t x_s^\theta C^\theta (D D^\theta)^{-1} dy_s - \frac{1}{2} \int_0^t x_s^\theta C^\theta (D D^\theta)^{-1} C x_s ds \mid \mathcal{Y}_t \right\} + \mathbf{E}\{R(\tilde{\theta}) \mid \mathcal{Y}_t\}, \end{aligned}$$

where $\#$ denotes the pseudoinverse and $R(\tilde{\theta})$ does not involve θ .

To implement the M-step we set the derivatives $\partial Q / \partial \theta = 0$. This yields

$$(5.4) \quad A = \mathbf{E} \left\{ \int_0^t dx_s x_s^\theta \mid \mathcal{Y}_t \right\} \left(\mathbf{E} \left\{ \int_0^t x_s x_s^\theta ds \mid \mathcal{Y}_t \right\} \right)^{-1} = \hat{L}_t^\theta \hat{H}_t^{-1},$$

$$(5.5) \quad C = \mathbf{E} \left\{ \int_0^t dy_s x_s^\theta \mid \mathcal{Y}_t \right\} \left(\mathbf{E} \left\{ \int_0^t x_s x_s^\theta ds \mid \mathcal{Y}_t \right\} \right)^{-1} = \hat{J}_t^\theta \hat{H}_t^{-1},$$

where $\hat{H}_t$ and $\hat{L}_t \in \mathbb{R}^{m \times m}$ denote matrices with elements $\hat{H}_t^{ij} \stackrel{\Delta}{=} \mathbf{E}\{H_t^{ij} \mid \mathcal{Y}_t\}$ and $\hat{L}_t^{ij} \stackrel{\Delta}{=} \mathbf{E}\{L_t^{ij} \mid \mathcal{Y}_t\}$, $i, j \in \{1, \dots, m\}$. Also, $\hat{J}_t \in \mathbb{R}^{m \times n}$ denotes the matrix with elements $\hat{J}_t^{ij} \stackrel{\Delta}{=} \mathbf{E}\{J_t^{ij} \mid \mathcal{Y}_t\}$, $i = 1, \dots, m$, $j = 1, \dots, n$. The terms $\hat{H}_t^{ij}$, $\hat{L}_t^{ij}$, and $\hat{J}_t^{ij}$ are computed using Theorems 3.2, 3.8, and 3.5 together with the filters in Theorem 3.10. Thus we have a filter-based EM algorithm.

Remark. We have presented the EM algorithm for updating the parameters of a general state-space model. However, we have not addressed identifiability issues of our state-space model.

Indeed, identifiability and consistency of the ML estimator have been studied in special cases of our model (where our filter-based algorithm also applies). For example, in [9], A is of a known structure but parametrized by an unknown vector θ . Consistency of the ML estimators is proved in [16].

Estimation of B and D . First consider the tensor product of x_t with itself:

$$(5.6) \quad x_t x_t^\theta = x_0 x_0^\theta + \int_0^t x_s dx_s^\theta + \int_0^t dx_s x_s^\theta + \int_0^t B B^\theta ds.$$

Conditioning both sides of (5.6) on $\mathcal{Y}_t$, we have

$$(5.7) \quad \Sigma_t = \mathbf{E}\{x_0 x_0^\theta | \mathcal{Y}_t\} + \mathbf{E} \left\{ \int_0^t x_s dx_s^\theta + \int_0^t dx_s x_s^\theta \middle| \mathcal{Y}_t \right\} + B B^\theta t.$$

$\mathbf{E}\{x_0 x_0^\theta | \mathcal{Y}_t\}$ in (5.7) is the smoothed second moment and is given in terms of finite-dimensional statistics; see Theorem 12.11, section 12.4 in [8]. The components of the conditioned stochastic integrals in (5.7) are given by the filtered estimates of L_t^{ij} . Consequently, we have a procedure for estimating the matrix $B B^\theta$.

Similarly, consider the tensor product of y_t with itself:

$$(5.8) \quad y_t y_t^\theta = \int_0^t y_s dy_s^\theta + \int_0^t dy_s y_s^\theta + D D^\theta t.$$

This expression simply amounts to evaluating DD^θ in terms of the quadratic variation of y .

6. Conclusion and extensions. For linear Gaussian dynamics, new finite-dimensional filters have been derived which estimate integrals and stochastic integrals of the moments of the state variable. Used in the EM algorithm, these quantities provide maximum likelihood estimates of the parameters in the dynamics of the Kalman filter.

We mention two extensions of the results. First, similar filters can be derived for nonlinear systems with Benes-type nonlinearity. Details are presented in [21]. Second, techniques similar to those developed in this paper can be applied in the reconstruction of doubly stochastic autoregressive (AR) processes, as we now outline.

Assuming the scalar version ($m = n = 1$) of the state-space model (2.1), (2.2), consider the continuous-time doubly stochastic AR process $z_t \in \mathbb{R}$ defined as

$$(6.1) \quad dz_t = f(x_t) z_t dt + du_t, \quad z_0 = 1.$$

Here u is a scalar Brownian process independent of v and w , and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial function. Suppose our aim is to derive a filter for computing $\mathbf{E}\{z_t | \mathcal{Y}_t\}$.

In (6.1), z_t can be viewed as a continuous-time AR process with random coefficient $f(x_t)$, where x_t itself evolves according to another AR process. In [4] we derived finite-dimensional filters for such problems in discrete time. The above model can be viewed as a continuous-time version of [4]. The problem of reconstructing z_t when x_t is a finite state Markov chain (instead of a continuous-valued Gaussian process) has recently been studied in the context of maneuvering target tracking [20]. Therefore, (6.1) can be viewed as the continuous-valued analogue of the model in [20].

Let us compute $\mathbf{E}\{z_t | \mathcal{Y}_t\}$. Note that we can solve (6.1) explicitly for z_t ; this yields

$$z_t = \exp \left(\int_0^t f(x_s) ds \right) \left[1 + \int_0^t \exp \left(- \int_0^s f(x_r) dr \right) du_s \right].$$

Since u is independent of x and y , our aim can be re-expressed as follows: compute the filtered estimate

$$\mathbf{E}\{z_t|\mathcal{Y}_t\} = \mathbf{E}\left\{\exp\left(\int_0^t f(x_s) ds\right) |\mathcal{Y}_t\right\}.$$

Using techniques similar to those presented in this paper, it is possible to derive a finite-dimensional filter for $\mathbf{E}\{z_t|\mathcal{Y}_t\}$. Details are presented in [22].

Appendix A. Derivation of $Q(\theta, \tilde{\theta})$. To update the estimate from $\tilde{A}$ to A , we employ Girsanov's theorem and introduce the density

$$\begin{aligned} \frac{dP(A)}{dP(\tilde{A})} \Big|_{G_t} &= \exp\left(\int_0^t x_s^\theta (A^\theta - \tilde{A}^\theta)(BB^\theta)^\# dx_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t x_s^\theta (A^\theta - \tilde{A}^\theta)(BB^\theta)^\# (A + \tilde{A}) x_s ds\right), \end{aligned}$$

where $\#$ denotes the pseudoinverse. Then

$$\begin{aligned} (A.1) \quad \mathbf{E}\left\{\log \frac{dP(A)}{dP(\tilde{A})} \Big|_{G_t} |\mathcal{Y}_t\right\} &= \mathbf{E}\left\{\int_0^t x_s^\theta A^\theta (BB^\theta)^\# dx_s - \frac{1}{2} \int_0^t x_s^\theta A^\theta (BB^\theta)^\# A x_s ds |\mathcal{Y}_t\right\} \\ &\quad + R(\tilde{A}), \end{aligned}$$

where $R(\tilde{A})$ does not involve A .

Similarly, to update the estimate from $\tilde{C}$ to C , we again apply Girsanov's theorem and introduce the density

$$\begin{aligned} \frac{dP(C)}{dP(\tilde{C})} \Big|_{G_t} &= \exp\left(\int_0^t x_s^\theta (C^\theta - \tilde{C}^\theta)(DD^\theta)^{-1} dy_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t x_s^\theta (C^\theta - \tilde{C}^\theta)(DD^\theta)^{-1} (C + \tilde{C}) x_s ds\right). \end{aligned}$$

Consequently,

$$\begin{aligned} (A.2) \quad \mathbf{E}\left\{\log \frac{dP(C)}{dP(\tilde{C})} \Big|_{G_t} |\mathcal{Y}_t\right\} &= \mathbf{E}\left\{\int_0^t x_s^\theta C^\theta (DD^\theta)^{-1} dy_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t x_s^\theta C^\theta (DD^\theta)^{-1} C x_s ds |\mathcal{Y}_t\right\} + S(\tilde{C}), \end{aligned}$$

where $S(\tilde{C})$ does not involve C .

Adding (A.1) and (A.2) yields (5.3).

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