Internal Model-based Controller Design using Measured Costs and Gradients

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June 11, 1997

Abstract

We consider the problem of tuning a closed-loop controller for a continuous-time, linear, lumped, exponentially stable plant using measurements. With the aim of reducing the amount of computation and the effect of modeling error, a two-step tuning method is proposed based on an internal model controller structure. The first step solves a controller design problem formulated as a sequence of convex optimization problems in which the cost, constraint functions, and their gradients are evaluated directly from plant measurements. The designer comfortably specifies desired performance by adjusting weights of a cost function composed of time and frequency domain performance functionals. Using plant frequency response data, the second step solves a convex optimization problem for a plant model required to implement the controller. Model error bounds ensure closed-loop stability. An example illustrates the design procedure.

1 Introduction

Controller design is commonly performed using a plant model obtained from laws of physics or measurements. In one design method functional inequalities and possibly a cost functional specify desired performance, and solving a numerical optimization problem yields controller parameters (Zakian and Al-Naib 1973). Numerous controller parameterizations, performance functionals, and algorithms have been proposed for controller design using numerical optimization. The work of Polak *et al.* (1984) and Boyd and Barratt (1991) and the references therein summarize the developments. The design method described in this paper uses numerical optimization and a controller parameterization, originated in Salcudean (1986), which leads to convex problems. The reader is referred to Boyd and Barratt (1991) for an thorough account of the benefits of convexity. Our approach uses an internal model controller, which depends on a plant model and a Youla parameter, and is therefore divided into two steps. In the first step, an optimal controller or Youla parameter Q_{opt} is obtained by solving a sequence of convex optimization problems where performance functionals and their gradients are approximately estimated directly from *measurements*. Indeed, many useful performance functionals can be approximated without a plant model provided enough accurate measurements are available. In the second step, a "control-oriented" plant model $\hat{P}_{opt}$ is obtained minimizing a cost which allows us to guarantee closed-loop stability assuming certain knowledge about plant. We expect that in certain applications where enough accurate measurements are available our approach will not only reduce the amount of computation during optimization but also, by eliminating the dependence of Q_{opt} on a possibly inaccurate plant model, yield higher performance relative to a model-based approach. However, the reader is warned that in general we make no guarantee about performance.

Our approach is related to a number of controller design methods that have appeared in the literature. For example, the benefits of the internal model structure and the use of experimental data for stability prediction have been noted in Gustafson and Desoer (1983). Although these authors use an internal model controller, non convex optimization problems result from the choice of parameterizations for both Q and the controller. Other uses of measurements to compute performance functionals and gradients of multi-objective optimization problems for tuning a PID controller appear in Seaman *et al.* (1991). Here the PID structure could conceivably limit the achievable performance since the optimization problems solved are not necessarily convex.

The identification step of our design relies on work done in Helmicki *et al.* (1992). These authors propose a two step method which approximates exponentially stable plants with sums of all-pass transfer functions and provides worst-case modeling error bounds. The first step obtains an $\mathbf{L}_\infty$ approximation of the plant composed with the bilinear transformation by calculating a truncated Fourier series of a linear spline interpolating the plant's frequency response data. The second step solves a Nehari problem to calculate the best $\mathbf{H}_\infty$ approximation to the Fourier series obtained in the first step. Our method makes use of the error bounds developed in Helmicki *et al.* (1992) to ensure stability. However, we obtain the model differently using convex optimization to directly minimize the $\mathbf{H}_\infty$ norm of the weighted plant model error.

Since our design of Q_{opt} affects our plant model $\hat{P}_{opt}$, our work is also related to iterative control design schemes which alternate between model based control design and identification to progressively increase closed-loop performance. Iterative methods have been proposed in for example Schrama (1992), Bayard *et al.* (1992), and Lee *et al.* (1993). In Bayard *et al.* (1992) the control design step solves a weighted mixed-sensitivity $\mathbf{H}_\infty$ problem. In the second step a plant model minimizes a p-norm of the model error weighted by the internal model parameter, obtained in the previous step, on a grid of frequency points. The resulting procedure is shown by example to provide robust performance. Our approach does not use iteration and we make no claims about robust performance.

The paper is organized as follows. Section 2 establishes some notation, describes the closed-loop system considered and the controller parameterization used. Section 3 explains the control step of the tuning procedure. Section 4 presents the identification step. Section 5 gives an example of the tuning procedure, and Section 6 concludes the paper.

2 Closed-loop system and controller parameterization

The unity-feedback system considered is shown in Figure 1, where P is a plant, K is a controller, r a reference signal, y a system output, d a disturbance input, m a measurement noise, e an error signal, and u a controller output. Our attention is restricted to plants and controllers which are linear, time-invariant, lumped, single-input, single-output, and operate in continuous-time. We use H_{ba} to denote the closed-loop transfer function from signal a

to signal b. The "real" plant with unknown transfer function P (although in Section 3 we shall assume further knowledge about P), is assumed strictly proper and exponentially stable. Letting $\Re s$ denote the real part of the complex number s, we denote $\mathbf{H}_\infty$ to be the space of complex-valued functions $H(s)$ of a complex variable whose $\mathbf{H}_\infty$ norm, defined as $\|H\|_\infty = \sup_{\Re s>0} |H(s)|$ is bounded. The space $\mathbf{RH}_\infty$ is the subspace of real-rational functions in $\mathbf{H}_\infty$. We use $\mathbf{H}_2$ to denote the space of complex-valued functions $H(s)$ whose $\mathbf{H}_2$ norm, defined as $\|H\|_2 = \left(1/(2\pi)\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega\right)^{1/2}$, is bounded. The space $\mathbf{RH}_2$ is the subspace of real-rational functions in $\mathbf{H}_2$.

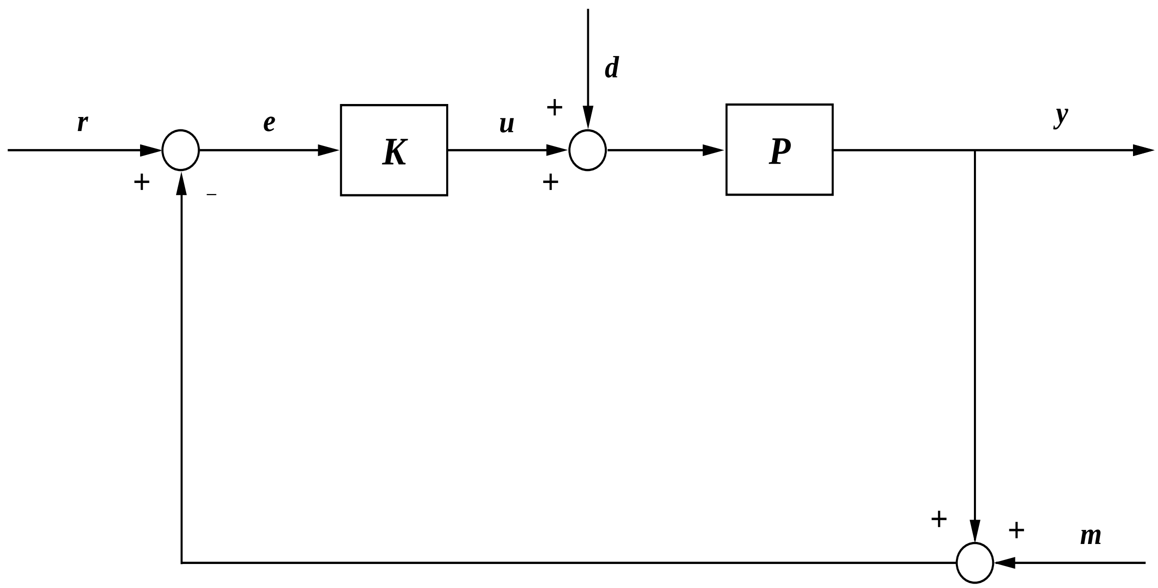

Figure 1: The closed-loop system.

The structure of the controller, called Internal Model Control in process control (Morari and Zafiriou 1989), is shown in Figure 2. The expression for K is

$$K(\hat{P}, Q) = Q(1 - \hat{P}Q)^{-1}, \tag{1}$$

where $Q, \hat{P} \in \mathbf{RH}_\infty$, $\hat{P}$ is a stable model of the plant, and Q is a free design parameter. The set of all stabilizing controllers for P is given by $\{K(P,Q) | Q \in \mathbf{RH}_\infty\}$ (Boyd and Barratt 1991). Hence, provided P is known and using Q as a design parameter the structure (1) limits our search to useful controllers. Although Q in (1) belongs to an infinite-dimensional space of functions, we cannot numerically optimize over all $\mathbf{RH}_\infty$ functions. Hence, we choose to approximate $\mathbf{RH}_\infty$ by some finite-dimensional linear subspace spanned by functions Q of

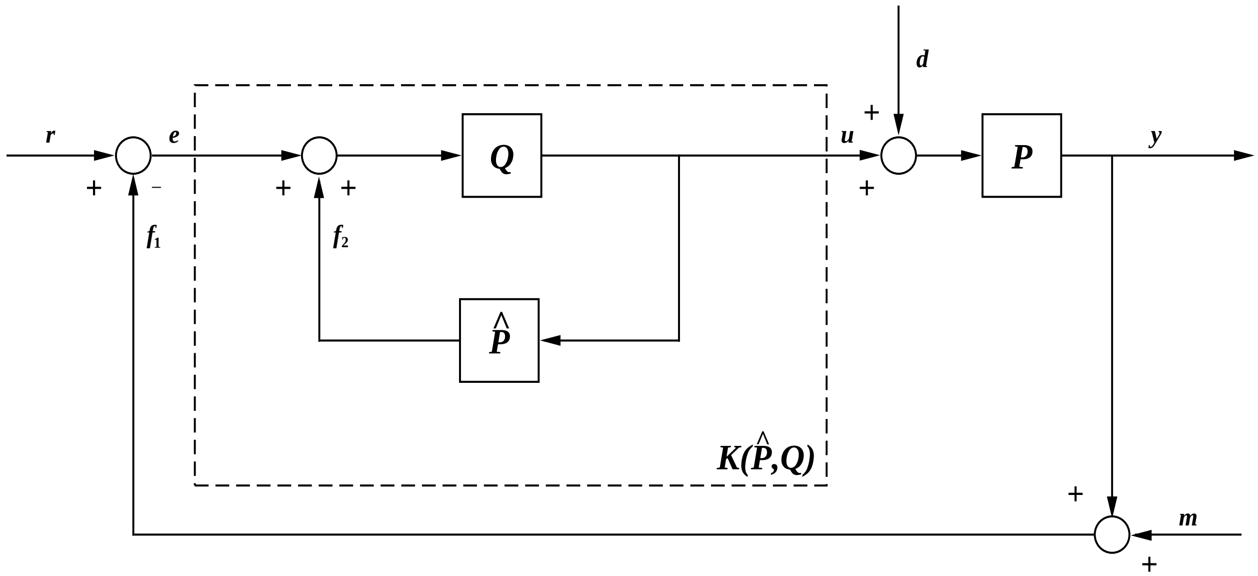

Figure 2: The IMC controller $K(\hat{P}, Q) = Q(1 - \hat{P}Q)^{-1}$.

the form

$$Q(x, s) = \sum_{k=1}^{N} x_k B_k(s), \tag{2}$$

where B_k are stable basis functions, and where $x_k \in \mathbf{R}$, $k \in \{1, \ldots, N\}$ are the controller parameters. For example, suppose $B_k(s) = (\frac{s-p}{s+p})^{k-1}, p > 0$ then the sets $\{Q(x,s) | x \in \mathbf{R^N}, N \text{ a positive integer}\}$ are dense in the Banach space $(\mathbf{RH}_\infty, \| \cdot \|_\infty)$ implying that any element of $\mathbf{RH}_\infty$ can be approximated arbitrarily well, at all frequencies, by sums of the form (2) provided N is large enough (Salcudean 1986). Although it is conceivable that N must be very large to obtain a good approximation, we believe this not to be a serious drawback in view of the simple structure of Q and the sophisticated DSP technology currently available.

If the plant is unstable and can be stabilized by a stable controller K_0, then the internal model controller takes the form

$$K_u(\hat{P}, Q) = K_0 + Q(1 - R(\hat{P})Q)^{-1}, \tag{3}$$

where $Q \in \mathbf{RH}_\infty$, and $R(\hat{P}) = \hat{P}(1 + K_0\hat{P})^{-1}$ (Maciejowski 1989). The fact that the second term in (3) has the same form as (1) with R substituted for $\hat{P}$ allows us to extend our approach to strongly stabilizable plants.

3 Control design step

3.1 Computing performance functionals from plant measurements

Performance functionals are nonnegative real-valued convex functions of the controller parameter x quantifying some aspect of closed-loop performance. Because of the parameterizations chosen in (1) and (2), many important performance functionals are convex in x. Examples of such functionals include the $\mathbf{H}_\infty$, $\mathbf{H}_2$ system norms of closed-loop transfer functions, and signal norms of closed-loop responses for particular inputs. Thorough discussion of convex performance functionals is in Boyd and Barratt (1991) and the subset of these functionals amenable to the tuning method is presented in Lynch (1993). This paper, for sake of brevity, considers only a few of the functionals in Lynch (1993).

An ellipsoid algorithm, presented in Grötschel *et al.* (1988), is used to solve the optimization problems in this paper. At each iteration, this algorithm requires the value of the cost and constraint functionals and their subgradients. A subgradient of a performance functional ϕ at x_0 is any $\zeta \in \mathbf{R}^\mathbf{N}$ satisfying

$$\phi(x) \geq \phi(x_0) + \zeta^T(x - x_0), \qquad \forall x \in \mathbf{R}^\mathbf{N}. \tag{4}$$

The functional ϕ need not be differentiable for a subgradient to exist and at least one subgradient exists if ϕ is convex.

We illustrate how measurements are used to compute two performance functionals and their subgradients. First, we consider a step response functional

$$\phi_{st}(x) = \max_{t \in [0, t_{max}]} \max\{\xi(x,t) - \alpha(t), \beta(t) - \xi(x,t), 0\}, \tag{5}$$

where ξ is the unit step response of an entry of the closed-loop transfer matrix, and the functions α and β are piecewise continuous upper and lower bound functions chosen by the designer. Appropriate α and β allow the designer to meet specifications involving overshoot, rise-time, "steady-state" error, and settling-time of closed-loop step responses. Letting t_0 be any time at which a maximum occurs in (5), a subgradient ζ_{st} of ϕ_{st} is

$$\zeta_{st}(x) = \begin{cases} \begin{bmatrix} \mathbf{L}^{-1}\left(\frac{H_1(s)}{s}\right)(t_0) \\ \vdots \\ \mathbf{L}^{-1}\left(\frac{H_N(s)}{s}\right)(t_0) \end{bmatrix}, & \phi_{st}(x) = \xi(t_0, x) - \alpha(t_0) > 0 \\ \begin{bmatrix} -\mathbf{L}^{-1}\left(\frac{H_1(s)}{s}\right)(t_0) \\ \vdots \\ -\mathbf{L}^{-1}\left(\frac{H_N(s)}{s}\right)(t_0) \end{bmatrix}, & \phi_{st}(x) = \beta(t_0) - \xi(x, t_0) > 0 \\ 0, & \phi_{st}(x) = 0 \end{cases}, \tag{6}$$

where H_k is the partial derivative of H (any closed-loop transfer function) with respect to x_k, and $\mathbf{L}^{-1}$ is the inverse Laplace transform operator.

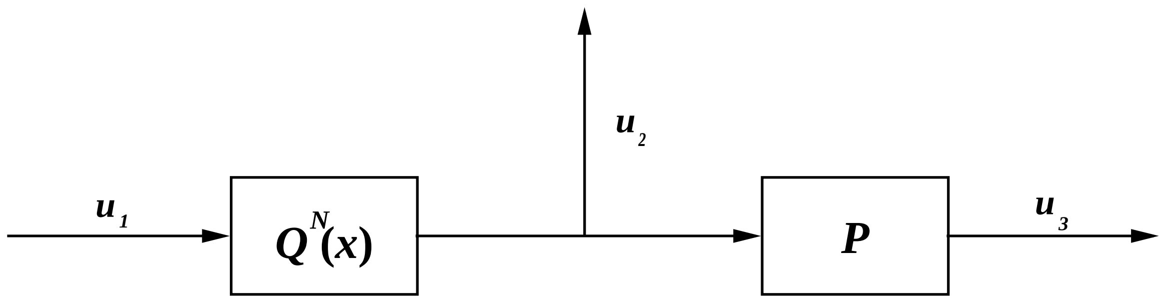

Figure 3: Open-loop system used to evaluate performance functionals and their subgradients using measurements.

Assuming noise-free measurements, we use open-loop system shown in Figure 3 and the following steps to evaluate ϕ_{st} for all entries of the closed-loop transfer matrix. We perform Step 1 before optimization begins and Steps 2–5 at each iteration of the optimization.

1. Apply a step input at u_1 (Figure 3) with $Q = 1$ and measure the step response of P at u_3 for $0 \leq t \leq t_{max}$.

2. Apply a step at u_1 and measure the step responses of Q and PQ at u_2 and u_3 respectively for $0 \leq t \leq t_{max}$.

3. Apply the step response measured in Step 1 at u_1 and measure the step response of P^2Q at u_3 for $0 \leq t \leq t_{max}$.

4. Add the responses obtained in Steps 1–3, to obtain the closed-loop responses. For example, to evaluate the step response of $H_{yd} = P(1-PQ)$ (when $\hat{P} = P$) we subtract the response measured in Step 3 from that measured in Step 1.

5. Evaluate ϕ_{st} using (5), with ξ being a step responses obtained in Step 4. Determine a time t_0 at which a maximum occurs.

Assuming a non-zero subgradient exists (otherwise no computation is required), the following steps evaluate a subgradient of ϕ_{st}. We perform Steps 1–4 before optimization begins and Step 5 at each iteration of the optimization algorithm.

1. Let $k = 1$.

2. Set the kth controller parameter x_k to 1 and the remaining ones to zero, apply a step input to u_1 and measure the step responses of the basis function B_k and PB_k at u_2 and u_3 respectively for $0 \le t \le t_0$.

3. Apply the response measured at u_3 in Step 2 to u_1 with $Q = 1$ and measure the step response of $P^2 B_k$ at u_3 for $0 \le t \le t_0$.

4. Increase k by 1 and repeat Steps 2–3 until $k = N + 1$.

5. Using (6), where H is some closed-loop transfer function, compute a subgradient using the value of the vector of step responses measured in Steps 2–3 at t_0.

In the case where the step responses are corrupted by noise the procedure to obtain ϕ_{st} and its subgradient could conceivably be performed a number of times and an average computed. Presently no attempt to study understand this situation is made leaving it an open issue.

Next we consider an "$\mathbf{H}_\infty$ norm" functional defined as

$$\phi_{Hinf}(x) = \sup_{\omega \in \mathbf{R}} |W_1(j\omega) H(x, j\omega) + W_2(j\omega)|, \tag{7}$$

where H is some closed-loop transfer function of the system shown in Figure 1, and W_1, W_2 are stable weighting transfer functions selected by the designer. Although (7) is not differentiable due to the supremum operation, it is convex and a subgradient is given by

$$\zeta_{Hinf}(x) = \frac{1}{\phi_{Hinf}(x)} \begin{bmatrix} \Re((W_1(j\omega_0)H(x,j\omega_0) + W_2(j\omega_0))^* W_1(j\omega_0)H_1(j\omega_0)) \\ \vdots \\ \Re((W_1(j\omega_0)H(x,j\omega_0) + W_2(j\omega_0))^* W_1(j\omega_0)H_N(j\omega_0)) \end{bmatrix}, \quad (8)$$

where ω_0 is any frequency such that $\phi_{Hinf}(x) = |W_1(j\omega_0)H(x,j\omega_0)+W_2(j\omega_0)|$, $\Re(z)$ denotes the real part of a complex number z, "*" denotes complex conjugation, and H_k denotes the partial derivative of H with respect to x_k (Boyd and Barratt 1991).

Since only a finite set of plant frequency response data is assumed available, we approximate (7) by taking a max over a grid of frequencies. The following shows that since P is stable and strictly proper it is possible to bound this approximation error for an "$\mathbf{H}_\infty$ norm" functional involving H_{yr}. Obtaining bounds for functionals and subgradients involving other closed-loop transfer functions proceeds in a similar manner. We define

$$P_{bl}(z) = P(\tau(z)),$$

where

$$\tau(z) = \lambda\frac{1-z}{1+z}, \quad \lambda > 0, \quad (9)$$

maps the closed unit disk onto the closed right half plane. The choice of the parameter λ is discussed below. We introduce two parameters ρ, M, and a function Φ, similar to those used in Helmicki *et al.* (1992) that give a rough description of the plant's frequency response. From Helmicki *et al.* (1992) there exist

- $\rho > 1$ such that P_{bl} is analytic in $\mathbf{D}_\rho = \{s \in \mathbf{C} | |s| < \rho\}$.
- $M > 0$ such that $\|P_{bl}\|_{\infty,\rho} \leq M$, where $\|P_{bl}\|_{\infty,\rho} = \sup_{z\in\mathbf{D}_\rho} |P_{bl}(z)|$.
- a continuous function $\Phi(\omega) > 0$ such that $|P(j\omega)| < \Phi(\omega), \forall\omega$ and $\lim_{\omega\to\infty}\Phi(\omega) = 0$.

We note that since ρ and M depend on P_{bl}, they depend on the choice of λ in (19).

Assuming we have n points of the plant's noisy frequency response at the frequencies $\omega_k = \lambda\tan(\pi k/l), k \in \{0,\ldots,n-1\}$, where l is an even integer greater than $2n$. Then we can compute a complex-valued linear spline P_{spbl} defined on $(-\pi, \pi]$ which interpolates the l points (θ_k, ν_k), $k \in \{1-(l/2), 2-(l/2), \ldots, l/2\}$ where

$$\nu_k = \begin{cases} P(j\omega_k) + \epsilon_k, & \text{when } k \in \{0, \ldots, n-1\} \\ 0, & \text{when } k \in \{n, \ldots, l/2\} \cup \{-n, \ldots, 1-(l/2)\} \\ P^*(j\omega_k) + \epsilon^*_{-k}, & \text{when } k \in \{-1, \ldots, 1-n\} \end{cases}, \tag{10}$$

$\theta_k = 2\pi k/l$, and P_{spbl} is linear on $(-\pi, (1-(l/2))2\pi/l]$ with $\lim_{\theta \to -\pi^+} P_{spbl}(\theta) = P_{spbl}(\pi)$. The terms ϵ_k in (10) satisfy $|\epsilon_k| \le \eta, \forall k$ and account for noise due to the finite amount of time used to obtain the frequency response data. It can be shown (Lynch 1993, Helmicki *et al.* 1992) that an upper bound on the error between the spline $P_{spline}(\omega) = P_{spbl} \circ 2\tan^{-1}(\omega/\lambda)$ defined on $\mathbf{R}$ and $P(j\omega)$ is

$$\sup_{\omega \in \mathbf{R}} |P(j\omega) - P_{spline}(\omega)| \le \max\left\{\frac{4M\pi}{l(\rho-1)}, \sup_{\theta \in [2\pi n/l, \pi]} \Phi(\lambda \tan(\theta/2))\right\} + \eta. \tag{11}$$

From (11) we note that as the number of data points n tends to infinity and if we choose l such that $l \ge 2n(1+n^{-\alpha}), \alpha > 0$, the second entry in the $\max\{\cdot\}$ tends to zero. Also, the first entry tends to zero since l tends to ∞ as n does. Choosing α large (respectively small) requires frequency response data over a large (respectively small) bandwidth. The size of the bandwidth can be controlled independently of the choice of λ. However, by choosing λ small we obtain finer spacing at lower frequencies for a given n and l. Typically, λ is taken as twice the highest plant frequency of interest. Using reasoning similar to that used to obtain (11) gives

$$\sup_{\omega \in \mathbf{R}} |P(j\omega)Q(j\omega) - P_{spline}(\omega)Q_{spline}(\omega)| \le \max\left\{\frac{4\bar{M}\pi}{l(\bar{\rho}-1)}, \sup_{\theta \in [2\pi n/l, \pi]} \bar{\Phi}(\lambda \tan(\theta n/2))\right\} + \|Q\|_\infty \eta, \tag{12}$$

where $Q_{spline}(\omega) = Q_{spbl} \circ 2\tan^{-1}(\omega/\lambda)$ and Q_{spbl} interpolates $Q_{bl}(z) = Q(\tau(z))$ at $\exp(j\theta_k)$. Since PQ belongs to $\mathbf{RH}_2$, being exponentially stable and strictly proper, there exist $\bar{\rho} > 1, \bar{M} > 0$ and a continuous function $\bar{\Phi}$ which satisfy $\|P_{bl}Q_{bl}\|_{\infty,\bar{\rho}} \le \bar{M}$, $|P(j\omega)Q(j\omega)| < \bar{\Phi}(\omega)$ with $\lim_{\omega \to \infty} \bar{\Phi}(\omega) = 0$ and $\bar{\Phi}(\omega) > 0, \forall \omega$. Computing $\bar{\rho}, \bar{M}$, and $\bar{\Phi}$ (which depend on the value of controller parameter x) is straightforward for a specific choice of basis B_k.

We use (11) and (12) to obtain an upper bound on the approximation in the discretization of (7). Letting $\phi_{Hinf1} = \|PQ\|_\infty$ and its approximation $\hat{\phi}_{Hinf1} = \max_{k \in \{0,\ldots,n-1\}} |P(j\omega_k)Q(j\omega_k)|$

we obtain

$$|\phi_{Hinf} - \hat{\phi}_{Hinf}| \leq \sup_{\omega \in \mathbf{R}} |P(j\omega)Q(j\omega) - P_{spline}(\omega)Q_{spline}(\omega)|. \tag{13}$$

From (11), (12), and (13) it is evident $|\phi_{Hinf} - \hat{\phi}_{Hinf}|$ can be made arbitrarily small with an appropriate frequency grid provided η is small.

The following details the approximate computation of ϕ_{Hinf} from the plant's frequency response data for all closed-loop transfer functions. We perform Steps 1 and 2 before optimization begins and Steps 3 and 4 at each iteration of the optimization algorithm.

1. Obtain the frequency response of the plant at the frequencies $\omega_k = \lambda \tan(\pi k/l), k \in \{0, \ldots, n-1\}$, where λ, n, and l are chosen to make the discretization error small enough.

2. Using the transfer function expressions for B_k, W_1, W_2, and the frequency response data of P, obtain the frequency responses of $W_1 P$, $W_1 P B_k$, and $W_1 P^2 B_k$, $k \in \{1, \ldots, N\}$, and W_2 at ω_k, $k \in \{0, \ldots, n-1\}$.

3. Calculate the frequency responses at ω_k of the weighted closed-loop transfer functions $W_1 P(1-PQ)+W_2$, $-W_1 P(1-PQ)+W_2$, $W_1(PQ-1)+W_2$, $W_1 PQ+W_2$, $-W_1 PQ+W_2$, $W_1 Q + W_2$, and $-W_1 Q + W_2$ using multiplication, addition, the responses obtained in Step 2, and the current controller parameter.

4. Evaluate an approximation to ϕ_{Hinf} by taking the maximum magnitude of the weighted closed-loop frequency responses obtained in Step 3. Set ω_0 to the lowest frequency at which the maximum occurs.

The following steps evaluate a subgradient of ϕ_{Hinf}.

1. At each iteration of the optimization algorithm, multiply the conjugate of the weighted closed-loop frequency responses at ω_0 (computed in Step 3 above) by its derivative at ω_0 (computed in Step 2 above).

2. A subgradient is computed using (8), where $(W_1(j\omega_0)H(x, j\omega_0))^* W_1(j\omega_0)H_k(j\omega_0)$ is the kth component of the vector of frequency responses computed in the previous step.

A number of remarks are in order. First, little computation is required to evaluate the performance functionals and their subgradients. This is in contrast to the more complicated and possibly ill-conditioned computation required to evaluate the performance functional from a model. A thorough comparison of computational costs is in Lynch (1993). Secondly, although using measurements to perform optimization is vulnerable to measurement noise and "discretization error" in the functionals and gradients, it is not affected my modeling error. Lastly, a similar but more cumbersome procedure can be used to compute functionals and subgradients when P is strongly stabilizable. In this case the plant in the open-loop system in Figure 3 is replaced by the closed-loop system $K_0 R(P)$.

3.2 Tuning procedure

We assume the design problem can be expressed by the inequalities

$$\left.\begin{array}{l} \phi_k(x) \leq a_k, \quad k \in \{1, \ldots, c\}, \\ \psi_k(x) \leq b_k, \quad k \in \{1, \ldots, d\}, \quad a_k, b_k \geq 0, \end{array}\right\} \tag{14}$$

where ϕ_k and ψ_k are performance functionals; the choice of ϕ_k is determined by the problem's flexible performance specifications and the choice of ψ_k determined by the hard constraints of the problem. The general form of the optimization problem solved is

$$\min_{x \in \mathbf{R}^N} \max\{\bar{\phi}(x) | \; \bar{\psi}(x) \leq 0\}, \tag{15}$$

where

$$\bar{\phi}(x) = \max\{\bar{\phi}_k(x), k \in \{1, \ldots, c\}\}, \qquad \bar{\psi}(x) = \max\{\bar{\psi}_k(x), k \in \{1, \ldots, d\}\}, \tag{16}$$

and where

$$\bar{\phi}_k = \frac{\phi_k - \delta_k}{\gamma_k - \delta_k}, \qquad \bar{\psi}_k = \frac{\psi_k - b_k}{b_k}, \quad \gamma_k, \delta_k, b_k \in \mathbf{R}. \tag{17}$$

Since hard constraints are inflexible, the normalization of the ψ_k in (17) is fixed. On the other hand, the normalization of the ϕ_k can be varied *throughout* the optimization to improve trade-off control and to ensure proper relative importance is assigned to the ϕ_k (Nye and Tits 1986). To understand this increased control we notice the normalization (17) provides two levels of designer satisfaction where the (normalized) objectives are equal. The two levels of designer satisfaction at which the normalized objectives are equal are 0 (respectively 1) when the objective reaches its "good" (respectively "bad") value δ_k (respectively γ_k). A linear relationship between designer satisfaction and normalized objective ensures the desired relative decrease in objectives is specified by the designer. A nonlinear relationship requires interactive adjustment of the weights when the design specifications are not satisfied. The following procedure is used to determine an x satisfying (14).

1. Select N, the basis functions B_k in (2), and initialize the controller parameter to $x^0 = 0$. (The superscript 0 denotes the initial parameter for an optimization problem.)

2. If the design has hard constraints, solve

$$x_{opt} = \arg\min\{\bar{\psi}(x),\ x \in \mathbf{R}^N\}.$$

   Stop if $\bar{\psi}(x_{opt}) > 0$ or the hard constraints are not achievable. Stop if $\bar{\psi}(x_{opt}) \leq 0$ and there are no soft constraints.

3. Let $x^0 = x_{opt}$ and solve

$$x_{opt} = \arg\min\{\bar{\phi}(x) | \bar{\psi}(x) \leq 0,\ x \in \mathbf{R}^N\},$$

   where initially $\delta_k = a_k,\ \gamma_k = 2a_k,\ k \in \{1, \ldots, c\}$. Stop if $\bar{\phi}(x_{opt}) \leq 0$ or all soft constraints are met.

4. Reassess the "good" and "bad" weights. Increase (respectively decrease) δ_k (respectively γ_k) of the objective we want to increase. Decrease (respectively increase) δ_k (respectively γ_k) of the objective we want to decrease. Repeat Step 3 with $x^0 = x_{opt}$ until acceptable performance or the best trade-off between objectives is achieved. Stop if acceptable performance cannot be found.

We make several remarks. First, if the procedure stops with undesirable performance in Step 4 or if no feasible solution is obtained in Step 2 then we repeat the procedure with more controller parameters. If increasing the number of parameters does not improve performance, then the procedure stops. Secondly, it is not always necessary to find the optimal solution x_{opt}. For example, if during Step 3 the soft constraints are not met but satisfactory performance is observed then optimization can be terminated. As well, optimization can be stopped if performance does not improve after many iterations. Finally, some judgment is required when performing the above procedure. If we design for extremely high performance then small errors in the plant model to be obtained in Section 4 may greatly affect closed-loop performance.

4 Identification step

This section presents an optimization-based identification method which follows work in Helmicki *et al.* (1992) in that a similar cost is minimized. However, unlike their approach we use *convex optimization* to approximately minimize

$$\|Q_{opt}(P - \hat{P})\|_\infty, \tag{18}$$

over all $\hat{P}$ in $\mathbf{RH}_\infty$, where $Q_{opt}(s) = Q(x_{opt}, s)$ is the optimal design parameter obtained in the control design step described in Section 3. (We have assumed the plant is strictly proper and could therefore optimize over $\mathbf{RH}_2 \subset \mathbf{RH}_\infty$. However, this would complicate the derivation of the error bound.) The reason for weighting the model error by Q_{opt} makes the identification dependent on desired closed-loop performance. For example, consider the expression for H_{yr} in terms of P, $\hat{P}$ and Q_{opt},

$$H_{yr} = (1 - Q_{opt}(\hat{P} - P))^{-1} P Q_{opt}.$$

Small errors in $\|Q_{opt}(\hat{P} - P)\|_\infty$ mean that H_{yr} is close to PQ_{opt}, the desired closed-loop transfer function, regardless of whether $\|\hat{P} - P\|_\infty$ is small or not.

As in Section 3 we assume the noisy frequency response data (10) is available and approximately minimize (18) by solving the convex problem

$$v_{opt} = \arg\min \left\{ \max \left\{ \left| Q_{opt}(j\omega_k)(\hat{P}^F(v, j\omega_k) - P(j\omega_k)) \right| \mid v \in \mathbf{R}^F, k \in \{0, \ldots, n-1\} \right\} \right\},$$

where $\hat{P}^F$ denotes the plant model

$$\hat{P}^F(v, s) = \sum_{k=1}^{F} v_k \left(\frac{\lambda - s}{\lambda + s} \right)^{k-1}, \quad \lambda > 0, \tag{19}$$

and v_k are model parameters. Using ρ, M, and Φ introduced in Section 3, an upper bound on the error can be computed:

$$\|P - \hat{P}_{opt}\|_\infty \leq \frac{M}{\rho^F(\rho - 1)} + \max\left\{\frac{4M\pi}{l(\rho - 1)}, \sup_{\theta \in [2\pi n/l, \pi]} \Phi(\lambda \tan(\theta/2))\right\} + \eta, \qquad (20)$$

where $\hat{P}_{opt}(s) = \hat{P}(v_{opt}, s)$. The derivation of (20) in Lynch (1993) and follows work in Helmicki *et al.* (1992). It is important to note, as with the bound (11), that as the number of data points n tends to infinity and if l is chosen such that $l \geq 2n(1 + n^{-\alpha}), \alpha > 0$, the second entry in the $\max\{\cdot\}$ in (20) tends to zero. Also, the first entry tends to zero since l tends to ∞ as n does. As well, as we increase F, the number of model parameters, the first term in (20) tends to zero. Evidently, stability of the closed-loop system can be guaranteed if the upper bound on the model error is less than $1/\|Q_{opt}\|_\infty$ and the assumed knowledge about the plant is correct.

For strongly stabilizable plants we minimize $\|Q_{opt}(R(P) - R(\hat{P}))\|_\infty$ over all $R(\hat{P})$ in $\mathbf{RH}_\infty$. The reason for minimizing this functional can be seen by writing the entries of the closed-loop transfer matrix as functions of $R = R(P)$, $\hat{R} = R(\hat{P})$, K_0, and Q. For example, consider $H_{yr} = (1 + Q(\hat{R} - R))^{-1}R(K_0 + Q - \hat{R}K_0Q)$ from which it is evident that making $Q(\hat{R} - R)$ small makes the closed-loop transfer functions approximately equal to those used to design Q_{opt}.

5 Example

This section presents a design example for the plant

$$P(s) = \frac{9}{(s + 1)(s^2 + .8s + 9)}.$$

The design specifications are

- a well dampened output step response which tracks a reference step input; and
- a positive gain margin greater than 10dB.

The performance functionals are

$$\begin{aligned} \phi_1(x) &= \|W_d H_{yd}\|_2^2 + \|H_{ym}\|_2^2 \\ \phi_2(x) &= \|W_d H_{ud}\|_2^2 + \|H_{um}\|_2^2, \end{aligned}$$

where

$$W_d(s) = \frac{1.23s^2 + 1.23s + 10.6}{s^3 + 2.00s^2 + 10.6s + .0212}. \tag{21}$$

Since $\|H_{um}\|_2 = \|Q\|_2$ must be finite, Q must belong to $\mathbf{RH}_2$. Hence, we take $B_k(s) = \sqrt{2p}\frac{(s-p)^{k-1}}{(s+p)^k}$, $p = 1$ which are the Laplace transforms of the Laguerre functions. The norms of $W_d H_{yd}$, H_{ym}, and $W_d H_{ud}$ are estimated using a trapezoidal integration rule and plant frequency response data obtained at the frequencies $\omega_k = \lambda \tan(k\pi/l)$, $k \in \{0, \ldots, n-1\}$, $n = 4000$, $l = 8002$, $\lambda = 10/\pi$. This is data from DC to 11.6 rad/s. Since $H_{um} = Q$ and we have an expression for Q, we obtain $\|H_{um}\|_2$ by solving a Lyapunov equation. Choosing $N = 20$ controller parameters and after some adjustment of the design weights an optimal controller parameter x_{opt} is found such that the performance specifications are met for $\eta = .001$. The noise level in the frequency response data is increased to .01 and .1 and the optimization is performed again. Assuming rather accurate plant information $\rho = 1.14$, $\Phi(\omega) = \frac{10}{(1+(\omega/7))^3}$ and $M = .7$, and using (20)closed-loop stability is ensured for $\eta = .1, .01, .001$ with $F = 20$ model parameters. Table 1 gives the optimal values of the estimated approximate cost functionals, obtained from the open-loop system and therefore independent of $\hat{P}$, and the exact functionals measuring the performance of the final closed-loop system depending on Q_{opt}, P, and $\hat{P}_{opt}$. The weights on ϕ_2 were set to $\delta_2 = 250, \gamma_2 = 300$ for all noise levels. The weights on ϕ_1 are shown in Table 1.

Table 1: Result of the tuning method for three levels of noise.

η	δ_1	γ_1	$\hat{\hat{\phi}}_1^*$	$\hat{\hat{\phi}}_2^*$	$\max\{\hat{\hat{\phi}}_1^*, \hat{\hat{\phi}}_2^*\}$	$\bar{\phi}_1^*$	$\bar{\phi}_2^*$	$\max\{\bar{\phi}_1^*, \bar{\phi}_2^*\}$
.001	1	2	.0188	.0162	.0188	.0435	.2124	.2124
.01	1	2	.0041	.0614	.0614	.0235	.1885	.1885
.1	.5	4	.3107	.5148	.5148	.2187	-.0781	.2187

Figure 4 shows the tuned closed-loop step response for $K(\hat{P}_{opt}, Q_{opt})$ for the three noise levels and the plant's open-loop response. For $\eta = .1$ the step response approximately tracks the reference but is too oscillatory and our approach fails. The Bode plots of the open-loop gain $PK(\hat{P}_{opt}, Q_{opt})$ are shown in Figure 5 for the three levels of noise. The gain and phase margins for $\eta = .1, .01, .001$ are 14.5dB, 72 deg., 14.4 dB, 71.9 deg., 15 dB, and 66 deg. respectively. Finally. we note that an easy way to have ensured asymptotic tracking of steps would have been to include an integrator in K or equivalently to force $Q(0) = 1$. The constraint $Q(0) = 1$ is convex and could have been included to achieve the specification.

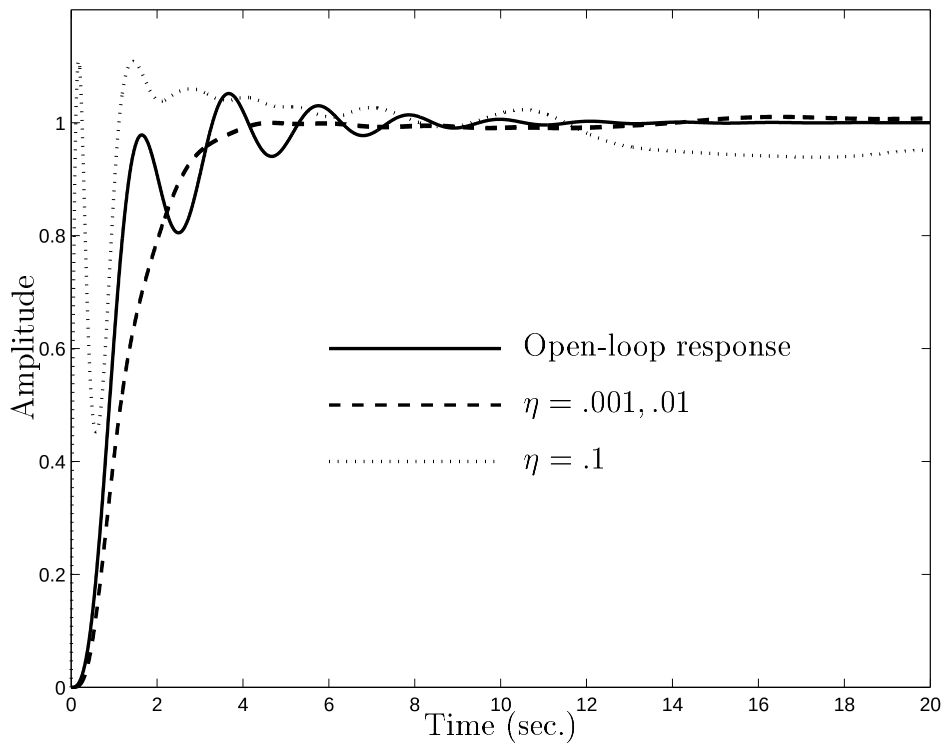

Figure 4: The output closed-loop step response of the tuned system for three levels of noise and the open-loop step response of the plant. The responses for η equal to .01 and .001 are indistinguishable.

6 Conclusion

In conclusion, this paper has presented a new optimization-based tuning method for linear compensators with internal model structure. Since such a controller structure requires both a design parameter Q and a plant model $\hat{P}$, our approach is divided into two steps each of which solves convex optimization problems. This convexity allows arbitrarily good approxi-

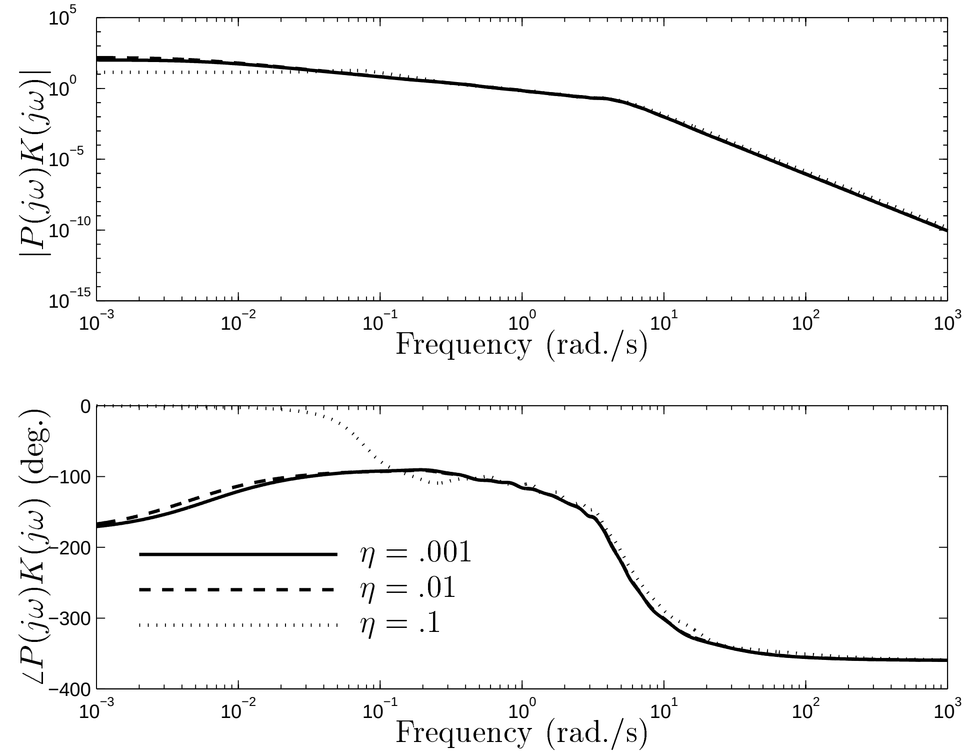

Figure 5: (a) and (b) show the Bode magnitude and phase plots respectively of the open-loop gain for the tuned system. The gain margin is approximately 15 dB for all three noise levels meeting the design specification. The phase margin is approximately 72 degrees for all three designs.

mation of the optimal solutions $Q_{opt}, \hat{P}_{opt}$ by solving a sequence of problems with increasing number of parameters. If enough measurements are available many performance functionals and their gradients can be approximated without a plant model and the optimization for Q can be performed on-line. As opposed to using a plant model, the use of measurements can reduce the amount of computation and do not introduce modeling error. The second step constructs $\hat{P}_{opt}$ minimizing the $\mathbf{H}_\infty$ norm of a model error weighted by Q_{opt}. Bounds on the modeling error ensure the closed-loop is stable. Future work consists of applying the tuning method to a real system. Such work would provide us a better understanding of the typical accuracy and number of measurement required to compute Q_{opt} and $\hat{P}_{opt}$.

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