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MULTIPLE OBJECTIVE CONTROL PROBLEMS VIA NONSMOOTH ANALYSIS

Zhongzhi Hu[1], S. E. Salcudean[1] and P. D. Loewen[2]

Department of Electrical Engineering[1]
and Institute of Applied Mathematics[2]
University of British Columbia
Vancouver, B. C. Canada V6T 1Z4
E-mail: huz@ee.ubc.ca

Abstract: A control system design problem involves making trade-offs among multiple competing objectives. This paper studies two multiple objective control problems via nonsmooth analysis. First, a new minimax solution approach to the multiple objective linear-quadratic optimal control problem is presented. The solution to this problem is characterized by a set of coupled Riccati equations. Second, the solution to the SISO multi-disk $\mathcal{H}_\infty$ control problem is pursued. Optimality conditions, and, in special cases, either all-pass properties or optimal performance values are obtained.

Key words: Linear Quadratic Regulators, $\mathcal{H}_\infty$ Control, Multiobjective Optimizations, Nonsmooth Analysis.

1. INTRODUCTION

In a single objective optimal control problem such as the $\mathbf{LQG}$ or the $\mathcal{H}_\infty$ problem, competing criteria are combined into a single objective function and performance tradeoffs are accomplished by adjusting the weighting functions to reflect the relative importance of performance and robustness specifications. This can be time consuming. A better approach would be to formulate and solve multiple objective control problems. This paper solves two multiple objective control problems via nonsmooth analysis (Clarke, 1989).

First, the multiple objective linear-quadratic optimal control problem, where the functional to be minimized is the maximum of several quadratic performance indices, is studied. Various theoretical results have been obtained by Medanic and Andjelic (1971), Makila (1989), Khargonekar and Rotea (1991) and Li (1990), to name a few. In (Khargonekar and Rotea, 1991; Makila, 1989), the set of noninferior solutions for multiple-objective linear-quadratic control problems is characterized, while in (Medanic and Andjelic, 1971; Li, 1990), the minimax solution is obtained from the set of noninferior solutions by searching weighting coefficients through either convex approximation or an equalizer strategy. In this paper, a new solution approach via nonsmooth analysis is proposed and the solution to this problem is obtained directly.

Second, the general multiple objective $\mathcal{H}_\infty$ control problem, also referred to as the multi-disk $\mathcal{H}_\infty$ control problem, is studied. This problem requires finding a stabilizing controller such that the maximum of several $\mathcal{H}_\infty$ norms of transfer functions of interest is minimized. In complete generality, multi-disk $\mathcal{H}_\infty$ problems have only been solved numerically by convex optimization (see, for example, Boyd and Barratt, 1991; Rotea and Prasanth, 1994) and are hard to solve explicitly. However, some qualitative properties of the optimum have been reported (Helton and Howe, 1986; Holohan and Safonov, 1992). Holohan and Safonov (1992) studied the nominal loop shaping problem for SISO systems, which is transformed into a two-disk $\mathcal{H}_\infty$ problem. They used function space duality theory to obtain bounds and all-pass properties. While the results obtained here are similar to the results of Holohan and Safonov (1992), the problem presented here is more general and the solution is simpler.

The rest of the paper is organized as follows: in the next section, some convex and nonsmooth analysis preliminaries are given; in Section 3 the solution to the multiple objective linear-quadratic optimal control problem is presented; in Section 4, the multi-disk $\mathcal{H}_\infty$ control problem is formulated and solved; and finally some concluding remarks are offered in Section 5.

2. SOME PRELIMINARIES

We shall be making use of the following results on optimality conditions and subdifferentials of max-type functions in nonsmooth analysis (Clarke, 1989). Let Y be a Banach space and C be a nonempty subset of Y. The subdifferential of the convex function f at the point y is denoted by $\partial f(y)$ and the normal cone of the convex set $C \subset Y$ at y by $N_C(y)$. $\underline{N} \triangleq \{1, 2, ..., N\}$. The unit simplex in R^m is denoted by $\mathbf{\Sigma_m} \triangleq \left\{ \mu \in R^m : \mu_i \geq 0, \sum_{i=1}^{m} \mu_i = 1 \right\}$.

Lemma 1 (Clarke, 1989): Consider the optimization problem

$$\mathbf{OP}: \ \min\{f(y) : y \in C \subset Y\}, \tag{1}$$

where $C \subset Y$ is closed and convex and $f: \ U \longrightarrow R$ is finite-valued and convex on some open convex set $U \supseteq Y$. Then $\overline{y}$ is the solution of $\mathbf{OP}$ if and only if

$$0 \in \partial f(\overline{y}) + N_C(\overline{y}). \tag{2}$$

Lemma 2 (Clarke, 1989): Let $U \subset Y$ be an open convex set. Suppose $f_i : U \longrightarrow R, \ i \in \underline{N}$, is a finite collection of functions each of which is finite-valued and convex on U. Define $f(y) = \max\{f_i(y) : i \in \underline{N}\}$, and denote by $I(y)$ the set of "active" indices i for which $f_i(y) = f(y)$. Then the function f is convex on U and its subdifferential obeys

$$\begin{aligned}\partial f(y) &= \mathrm{co}\{\partial f_i(y), i \in I(y)\} \\ &= \left\{ \sum_{i \in I(y)} \lambda_i \xi_i : \xi_i \in \partial f_i(y), \lambda_i \geq 0, \sum_{i \in I(y)} \lambda_i = 1 \right\}\end{aligned} \tag{3}$$

In particular, at any point y where each f_i is differentiable, we may take $\xi_i = \nabla f_i(y)$ in (3).

Lemma 3 (Clarke, 1989): Let Y be a Banach space and f_θ be a family of functions on Y parametrized by $\theta \in T$, where T is a compact topological space. Assume

1. Each $f_\theta(y)$, $\theta \in T$, is Lipschitz at y, and $\{f_\theta(y) : \theta \in T\}$ is bounded;
2. $f_\theta(y)$ is continuous as a function of θ and convex as a function of y.

Define a function $f : Y \rightarrow R$ via

$$f(y) = \max_{\theta \in T} \{f_\theta(y)\}. \tag{4}$$

Then the subdifferential of f is given by

$$\partial f(y) = \left\{ \int_T \partial f_\theta(y) \mu(d\theta) : \mu \in P[M(y)] \right\}. \tag{5}$$

Here, $M(y) \triangleq \{\theta \in T : f_\theta(y) = f(y)\}$ denotes the "active set" and for any subset S of T, $P[S]$ signifies the collection of probability Radon measures supported on S.

3. MULTIOBJECTIVE LQ CONTROL PROBLEMS

Consider the linear system described by

$$\dot{x} = Ax + Bu, \ x(0) = x_0, \tag{6}$$

and the $N (N \geq 2)$ performance indices defined by

$$J_i(u) = \frac{1}{2} x^T(t_f) F_i x(t_f) + \frac{1}{2} \int_0^{t_f} \left(x^T Q_i x + u^T R_i u \right) dt, \tag{7}$$

for $i \in \underline{N}$, where $x \in R^n$ is the state vector, $u \in R^p$ is the control vector, t_f is the time horizon, and F_i, Q_i and R_i are symmetric matrices of appropriate dimensions. Assume throughout that the matrices F_i and Q_i are positive semidefinite while the matrices R_i are positive definite, and that each J_i has a finite minimum value. The multiple objective linear-quadratic control problem is formulated as follows:

$$\begin{aligned}\mathbf{MOLQP}: \ & \min_u J(u) \triangleq \min_u \max_{i \in \underline{N}} \{J_i(u)\} \\ & s.t. \quad \dot{x} = Ax + Bu, \ x(0) = x_0.\end{aligned} \tag{8}$$

Before we go further note that the overall objective function, $J(u)$, is not necessarily differentiable and $\mathbf{MOLQP}$ is not in a form that can be handled by the usual methods for $\mathbf{LQ}$ design. However $\mathbf{MOLQP}$ can be solved by using the tools of nonsmooth analysis.

Notice that $\mathbf{MOLQP}$ is equivalent to minimizing the function

$$\widetilde{J}(u, x, v) \triangleq \max\left\{ \widetilde{J}_i(u, x, v), i \in \underline{N} \right\} \tag{9}$$

over the affine subspace of $L^2 \times L^2 \times R^n$ defined by

$$\widetilde{S} = (0, \widetilde{x}, \widetilde{x}(t_f)) + S, \tag{10}$$

where

$$\widetilde{J}_i(u,x,v) \triangleq \frac{1}{2}v^T F_i v + \frac{1}{2}\int_0^{t_f} \left(x^T Q_i x + u^T R_i u\right) dt, \quad (11)$$

$$\widetilde{x}(t) = e^{At} x_0, \quad (12)$$

and S is the space defined as a subspace of $L^2 \times L^2 \times R^n$:

$$S \triangleq \left\{ \begin{array}{c} (u(.),x(.),v) : x(t) = \int_0^t [Ax(s) + Bu(s)]ds, \\ v = x(t_f) \end{array} \right\}. \quad (13)$$

It can be easily shown that $\widetilde{J}$ and $\widetilde{S}$ satisfy the conditions of Lemma 1. Consequently $(\overline{u}, \overline{x}, \overline{v})$ is the solution to **MOLQP** if and only if

$$0 \in \partial \widetilde{J}(\overline{u}, \overline{x}, \overline{v}) + N_{\widetilde{S}}(\overline{u}, \overline{x}, \overline{v}). \quad (14)$$

Now for any $\lambda \in \Sigma_N$, let $F_\lambda = \sum_{i=1}^N \lambda_i F_i$, $Q_\lambda = \sum_{i=1}^N \lambda_i Q_i$, and $R_\lambda = \sum_{i=1}^N \lambda_i R_i$. An optimal control can be obtained by the following theorem.

Theorem 1: Assume that all performance indices $J_i (i \in \underline{N})$ conflict with each other and have the same numerical value for the minimizing control. If some vector $\lambda \in \Sigma_N$ and matrix $P = P^T \in R^{n \times n}$ satisfy the coupled Riccati equations (16)-(18) below, then **MOLQP** has a solution of the form

$$\overline{u} = -R_\lambda^{-1} B^T P \overline{x}. \quad (15)$$

The equations are

$$\dot{P} + PA + A^T P + Q_\lambda - PBR_\lambda^{-1} B^T P = 0, P(t_f) = F_\lambda, \quad (16)$$

$$\begin{array}{l} \dot{W}_i + W_i\left(A - BR_\lambda^{-1}B^T P\right) + \left(A - BR_\lambda^{-1}B^T P\right)^T W_i \\ +Q_i + PBR_\lambda^{-1} R_i R_\lambda^{-1} B^T P = 0, W_i(t_f) = F_i, i \in \underline{N}, \end{array} \quad (17)$$

$$x_0^T (W_i(0) - W_j(0)) x_0 = 0, i,j \in \underline{N}. \quad (18)$$

Proof: By assumption and from Lemma 2, at $(\overline{u}, \overline{x}, \overline{v})$, the subdifferential of $\widetilde{J}$ is given by

$$\partial \widetilde{J}(\overline{u}, \overline{x}, \overline{v}) = \left\{ \sum_{i=1}^N \lambda_i \nabla \widetilde{J}_i(\overline{u}, \overline{x}, \overline{v}) : \lambda \in \Sigma_N \right\}, \quad (19)$$

with $\forall (u,x,v) \in L^2 \times L^2 \times R^n$,

$$\begin{array}{l} < \nabla \widetilde{J}_i(\overline{u}, \overline{x}, \overline{v}), (u,x,v) > \\ =< F_i \overline{v}, v > + \int_0^{t_f} (< Q_i \overline{x}, x > + < R_i \overline{u}, u >) dt. \end{array} \quad (20)$$

From (10),

$$N_{\widetilde{S}}(\overline{u}, \overline{x}, \overline{v}) = N_S(\overline{u}, \overline{x} - \widetilde{x}, \overline{v} - \widetilde{x}(t_f)), \quad (21)$$

and since S is a subspace of $L^2 \times L^2 \times R^n$,

$$N_S(\overline{u}, \overline{x} - \widetilde{x}, \overline{v} - \widetilde{x}(t_f)) = S^\perp, \quad (22)$$

where $S^\perp$ is the orthogonal complement of S given by Loewen (1993) as follows:

$$S^\perp = \left\{ \left(B^T p, \dot{p} + A^T p, -p(t_f)\right) : p \in AC^2([0,t_f]; R^n) \right\}. \quad (23)$$

Therefore, from (14) and (19) to (22), $(\overline{u}, \overline{x}, \overline{v})$ is the solution to **MOLQP** if and only if

$$0 \in \partial \widetilde{J}(\overline{u}, \overline{x}, \overline{v}) + S^\perp. \quad (24)$$

The inclusion (24) is equivalent to the existence of a vector λ and an absolutely continuous function p satisfying the following two conditions:

$$R_\lambda \overline{u} + B^T p = 0, \quad (25)$$

$$Q_\lambda \overline{x} + \dot{p} + A^T p = 0, \text{ and} \quad (26)$$

$$p(t_f) = F_\lambda. \quad (27)$$

If (16) has a solution $P = P^T$, then the choice

$$p(t) = P\overline{x}(t) \quad (28)$$

will satisfy (26) as a consequence of (6), and hence (15) will hold as a consequence of (25). Defining

$$\begin{array}{l} x^T(t) W_i(t) x(t) \\ = x^T(t_f) F_i x(t_f) + \int_t^{t_f} \left(x^T Q_i x + u^T R_i u\right) dt, \end{array} \quad (29)$$

substitution of the control (15) into the state equation (6) and the performance index (7) gives, after some manipulation, the following expression for the performance index:

$$J_i = \frac{1}{2} x_0^T W_i(0) x_0, i \in \underline{N}, \quad (30)$$

where $W_i(t)$ satisfies equation (17). The condition (18) follows easily from the assumption that all the performance indices $\{J_i, i \in \underline{N}\}$ have the same numerical value for the minimizing control. ■

Note that the active indices that are involved at the optimum can be identified beforehand through an exhaustive search. A better method to do this would be to use convex duality (Loewen et al., 1996).

From Theorem 1, we need to solve $(N+1)\times\frac{1}{2}n(n+1)+N\ (N\geq 2)$ equations, with the same number of unknown variables: $P=P^T, W_i=W_i^T\ (i\in\underline{N})$, and $\lambda_i(i\in\underline{N})$. It seems likely that for well-behaved systems, these equations will have a unique solution. There exist some numerical algorithms for the solution of the above coupled Riccati equations (Richter, 1987). The solution to $\mathbf{MOLQP}$ depends on the initial states via (18) and is an open-loop control optimal only for the particular initial conditions. This is different from the solution of the single-objective $\mathbf{LQ}$ optimal control problem, where one can always get a feedback control law. To illustrate this, a simple problem given in (Medanic and Andjelic, 1971) was solved by Theorem 1 for two different initial conditions.

Example 1 (Medanic and Andjelic, 1971):

$$J^*=\min_u\max_i\left\{\frac{1}{2}\int_0^\infty\left(x^TQ_ix+u^TR_iu\right)dt, i\in\underline{2}\right\},$$
$$s.t.\ \dot{x}=Ax+Bu,\ x(0)=x_0,\ \ \text{where}, \tag{31}$$

$$Q_1=\begin{bmatrix}2&1\\1&1\end{bmatrix},\ Q_2=\begin{bmatrix}1&-1\\-1&3\end{bmatrix},$$
$$R_1=2,\ R_2=1,$$
$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ \text{and } B=\begin{bmatrix}0\\1\end{bmatrix}. \tag{32}$$

The results are obtained by solving (15) to (18):

1. For $x_0=\begin{bmatrix}5\\2\end{bmatrix}$, $J^*=55.0208$, and the optimal control $u^*=[-1.000\quad -2.0690]x$.
2. For $x_0=\begin{bmatrix}-2\\5\end{bmatrix}$, $J^*=24.7723$, and the optimal control $u^*=[-1.000\quad -1.9310]x$.

4. MULTI-DISK $\mathcal{H}_\infty$ CONTROL PROBLEMS

In this paper, only multi-disk $\mathcal{H}_\infty$ problems for SISO systems are considered. The framework of general control systems as shown in Fig. 1 is used, where G is the generalized plant, which has absorbed all weighing functions, with two sets of inputs: the exogenous inputs $w=(w_1\ w_2\ ...\ w_{n_w})^T$, and control inputs u. The plant G also has two sets of outputs: the measured outputs y and the controlled outputs $z=(z_1\ z_2\ ...z_{n_z})^T$. K is the controller to be designed. It is assumed that G and K are real-rational and proper.

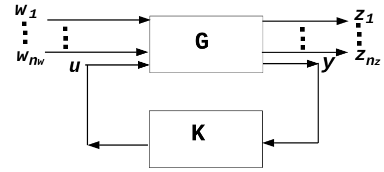

Figure 1 The multiple objective control problem

The multi-disk problem (P^0) is defined as follows: find a stabilizing controller K such that K minimizes the maximum of the $\mathcal{H}_\infty$ norms of the transfer functions $T_i\ (i\in\underline{m})$, i.e., solve

$$\left(P^0\right):\min_{\text{stabilizing } K}\max_{i\in\underline{m}}\left\{\|T_i(K)\|_\infty^2\right\}, \tag{33}$$

where $T_i\ (i\in\underline{m})$ denotes the closed-loop transfer function from an input $w_j\ (j\in\underline{n_w})$ to an output $z_k\ (k\in\underline{n_z})$.

By introduction of the Q-parametrization of the stabilizing controllers (Youla, et al., 1976), $T_i\ (i\in\underline{m})$ are affine transfer functions in free parameter $q\in\mathcal{H}_\infty$. So the multi-disk $\mathcal{H}_\infty$ problem for SISO systems (P^0) is equivalent to the following problem:

$$\left(P^1\right):\ \min_{q\in\mathcal{A}_0}\max_{i\in\underline{m}}\left\{\|a_iq-b_i\|_\infty^2\right\}, \tag{34}$$

where, a_i and $b_i\ (i\in\underline{m})$ are known stable transfer functions that can be obtained directly from the plant. Each of the transfer functions can be mapped from the left half plane to the unit disk by using bilinear transformation. Furthermore, for technical reasons, we reduce the space over which the minimization is performed in (34) to the Banach algebra $\mathcal{A}_0$ of $\mathcal{H}_\infty$ functions that are continuous on the unit circle (Garnett, 1981). Note that the problem (P^1) is convex and nondifferentiable. Nonsmooth analysis provides a powerful tool for problems with this structure.

Let $f(\cdot):\mathcal{A}_0\rightarrow R$ be given by

$$\begin{aligned}f(q)&=\max_{i\in\underline{m}}\left\{\|a_iq-b_i\|_\infty^2\right\}\\&=\max_{i\in\underline{m}}\left\{\max_{\theta\in T}\left\{|a_iq-b_i|^2\left(e^{j\theta}\right)\right\}\right\}\\&=\max_{\theta\in T}\max_{i\in\underline{m}}\left\{|a_iq-b_i|^2\left(e^{j\theta}\right)\right\}\\&=\max_{\theta\in T}f_\theta(q),\end{aligned} \tag{35}$$

where f_θ is a family of functions on $\mathcal{A}_0$ parametrized by $\theta\in T\overset{\triangle}{=}[0,2\pi]$, and is given by

$$f_\theta(q)=\max_{i\in\underline{m}}\left\{|a_iq-b_i|^2\left(e^{j\theta}\right)\right\}. \tag{36}$$

Clearly (P^1) is equivalent to

$$(P^2): \min_{q \in \mathcal{A}_0} f(q), \tag{37}$$

and the optimality condition for (P^2) is

$$0 \in \partial f(q). \tag{38}$$

The trivial case in which $a_i q - b_i = 0$ $(i \in \underline{m})$ have a solution in $\mathcal{A}_0$ is excluded. To get the optimality condition for (P^2), the subdifferential of f is computed by using Lemma 3. It can be verified that the conditions in Lemma 3 are satisfied. From Lemma 3, for any direction $h \in \mathcal{A}_0$,

$$< \partial f(q), h >= \left\{ \begin{array}{c} \int_T < \partial f_\theta, h > d\mu(\theta) : \mu \in P[S(q)], \\ h \in \mathcal{A}_0 \end{array} \right\}, \tag{39}$$

where, $S(q) \triangleq \{\theta \in T : f(q) = f_\theta(q)\}$. By Lemma 2, for any direction $h \in \mathcal{A}_0$,

$$< \partial f_\theta(q), h >= \left\{ \begin{array}{c} 2\mathrm{Re}\left\{ \sum_{i=1}^m [\alpha_i (a_i q - b_i)^* a_i](e^{j\theta})\, h(e^{j\theta}) \right\} : \\ \alpha(\theta) \in \Sigma_m, h(e^{j\theta}) \in \mathcal{A}_0 \end{array} \right\}. \tag{40}$$

From (38) to (40),

$$\begin{aligned} &\widehat{q} \quad \text{solves} \quad (P^2) \\ &\Rightarrow\ 0 \in \partial f(\widehat{q}) \\ &\Rightarrow\ \exists \mu(\theta) \in P[S(\widehat{q})] \ \text{such that} \ \forall h \in \mathcal{A}_0, \\ &0 = \int_T \mathrm{Re}\left\{ \sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] h \right\} \left(e^{j\theta}\right) d\mu(\theta). \end{aligned} \tag{41}$$

Since for a fixed h, $(-jh)$ also satisfies (41), therefore

$$0 = \int_T \left\{ \sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] h \right\} \left(e^{j\theta}\right) d\mu(\theta). \tag{42}$$

The following theorem gives the all-pass properties of the solution, and, in special cases, the optimal performance value.

Theorem 2: Assume that $a_i q - b_i = 0$ $(i \in \underline{m})$ have no solution in $\mathcal{A}_0$. If the multi-disk $\mathcal{H}_\infty$ problem $(m \geq 2)$ has a solution $\widehat{q} \in \mathcal{A}_0$, then either

(1) $\max\left\{ |a_i \widehat{q} - b_i|^2 \left(e^{j\theta}\right), i \in \underline{m} \right\} = \text{const.}$ $, \forall \theta \in T$, or

(2) if for each $i \in \underline{m}$, $a_i\left(e^{j\theta}\right) \neq 0$, $\forall \theta \in T$, then $\exists \widehat{\theta} \in S(\widehat{q})$, the active set corresponding to the optimal solution $\widehat{q}\left(e^{j\theta}\right)$, such that

$$\widehat{q}\left(e^{j\hat{\theta}}\right) = \sum_{i=1}^m \beta_i\left(\widehat{\theta}\right) \frac{b_i}{a_i}\left(e^{j\hat{\theta}}\right), \beta \in \Sigma_m. \tag{43}$$

Proof: We consider Case (1):

$$\sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \left(e^{j\theta}\right) \neq 0, \forall \theta \in T, \tag{44}$$

and Case (2): $\exists \widehat{\theta} \in S(\widehat{q})$ such that

$$\sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \left(e^{j\hat{\theta}}\right) = 0, \tag{45}$$

where, $\alpha\left(\widehat{\theta}\right) \in \Sigma_m$.

For Case (1), choose $h = e^{jn\theta}, n > 0$. Then (42) becomes

$$\int_T e^{jn\theta} \sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \left(e^{j\theta}\right) d\mu(\theta) = 0, n > 0. \tag{46}$$

From (Douglas, 1972), $\exists f\left(e^{j\theta}\right) \in \mathcal{H}_1$ such that

$$\sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \left(e^{j\theta}\right) d\mu(\theta) = f\left(e^{j\theta}\right) d\theta. \tag{47}$$

Since μ is a positive measure, and by the assumption, $\sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \left(e^{j\theta}\right) \neq 0, \forall \theta \in T$, f is a nonzero function in $\mathcal{H}_1$. Therefore the support of f is

$$\mathrm{supp}(f) \triangleq \mathrm{supp}(|f|) = T. \tag{48}$$

From (Rudin, 1966), since $d\mu \geq 0$ and $d\theta \geq 0$, (47) becomes

$$\left| \sum_{i=1}^m \left[\alpha_i (a_i \widehat{q} - b_i)^* a_i\right] \right| d\mu = |f|\, d\theta. \tag{49}$$

Therefore

$$\mathrm{supp}(\mu) = \mathrm{supp}(|f|) = T, \tag{50}$$

i.e., (1) of Theorem 2 holds. In this case, the "all-pass" property from single-objective $\mathcal{H}_\infty$ control theory extends to the multiobjective case.

For Case (2), since $a_i\left(e^{j\theta}\right) \neq 0, \forall \theta \in T$, from (45),

$$\sum_{i=1}^m \left[\alpha_i |a_i|^2 \left(\widehat{q} - \frac{b_i}{a_i}\right)\right] \left(e^{j\hat{\theta}}\right) = 0, \tag{51}$$

which immediately gives the desired result in (43) by choosing

$$\beta_i\left(e^{j\hat{\theta}}\right) \triangleq \frac{\alpha_i |a_i|^2}{\sum_{i=1}^m \alpha_i |a_i|^2} \left(e^{j\hat{\theta}}\right) \ (i \in \underline{m}). \ \blacksquare \tag{52}$$

From Theorem 2, the explicit form of the optimal performance value, denoted by γ, in case (2) can be obtained by solving following equations:

$$|a_i \widehat{q} - b_i| = \gamma, (i \in \underline{m}), \tag{53}$$

$$\widehat{q} = \sum_{i=1}^{m} \beta_i \frac{b_i}{a_i}, \tag{54}$$

$$\sum_{i=1}^{m} \beta_i = 1, \beta_i \geq 0, i \in \underline{m}. \tag{55}$$

This could be done by using symbolic math software. For example, for the two-disk problem,

$$\left(P^3\right): \min_{q \in \mathcal{A}_0} \max_{i \in \underline{2}} \left\{ \|a_i q - b_i\|_\infty^2 \right\}, \tag{56}$$

The similar result to that in (Holohan and Safonov, 1992) can be obtained by solving (53–55) via Maple (Char, et al., 1988).

Corollary 1: Assume that $a_i q - b_i = 0$, $(i \in \underline{2})$, have no solution in $\mathcal{A}_0$. If the two-disk $\mathcal{H}_\infty$ problem $\left(P^3\right)$ has a solution $\widehat{q} \in \mathcal{A}_0$, then either

(1) $\max \left\{ |a_i \widehat{q} - b_i|^2 \left(e^{j\theta}\right), i \in \underline{2} \right\} = \text{const.}\ , \forall \theta \in T,$ or

(2) if for each $i \in \underline{2}$, $a_i\left(e^{j\theta}\right) \neq 0$, $\forall \theta \in T$, then the optimal performance is given by $\left\| \frac{a_1 b_2 - a_2 b_1}{|a_1| + |a_2|} \right\|_\infty^2$.

Corollary 2: Assume that the multi-disk $\mathcal{H}_\infty$ problem $(m \geq 2)$ has a solution $\widehat{q} \in \mathcal{A}_0$. If $\max \left\{ |a_i \widehat{q} - b_i|^2, i \in \underline{m} \right\} = \gamma \ , \forall \theta \in T,$ and $\exists i \in \underline{m}$ such that the measure of the set $\left\{ \forall \theta \in T: \gamma = |a_i \widehat{q} - b_i|^2 \left(e^{j\theta}\right) \right\} \subsetneq T$ is not zero, then $\widehat{q}$ is of infinite order.

Proof: Notice that $a_i (i \in \underline{m})$ and $b_i (i \in \underline{m})$ are given real-rational transfer functions of finite order. ■

5. CONCLUDING REMARKS

Nonsmooth analysis has been shown to be a useful tool in the study of minimax-type optimal control problems. Within this framework, both multiobjective linear-quadratic control problems and multiobjective $\mathcal{H}_\infty$ control problems were studied. Necessary conditions or optimality conditions were given. However, finding the corresponding controllers either numerically or analytically still needs further research.

REFERENCES

Boyd, S. P. and C. H. Barratt (1991). *Linear Controller Design: Limits of Performance*, Prentice Hall.

Clarke, F. H. (1989). *Optimization and Nonsmooth Analysis*, Les publications CRM, Montreal.

Douglas, R. W. (1972). *Banach Algebra Techniques in Operator Theory*, Academic Press.

Char, B. W., et al. (1988). *Maple Reference Manual*, WATCOM Publications Limited.

Garnett, J. B. (1981). *Bounded Analytic Functions*, Academic Press.

Helton, J. W. and R. E. Howe (1986). A bang-bang theorem for optimization over spaces of analytic functions, *Journal of approximation theory*, **Vol. 47**, pp. 101–121.

Holohan, A. M. and M. G. Safonov (1992). Duality relations for the optimal two disk $\mathcal{H}_\infty$ problem, In *Proc. of ACC*, pp. 1844–1849.

Khargonekar P. P. and M. A. Rotea (1991). Multiple objective optimal control of linear systems: the quadratic norm case, *IEEE Transactions on Automatic Control*, **AC-36**, pp. 14–24.

Li, D. (1990). On the minimax solution of multiple linear-quadratic problems, *IEEE Transactions on Automatic Control*, **AC-35**, pp. 1153–1156.

Loewen, P. D. (1993). *Optimal Control via Nonsmooth Analysis*, Les publications CRM, Montreal.

Loewen, P. D., Z. Hu and S. E. Salcudean (1996). A new minimax solution approach for multiple objective linear quadratic control, in preparation.

Makila, P. M. (1989). On multiple criteria stationary linear quadratic control, *IEEE Trans. on Automatic Control*, **AC-34**, pp. 1311–1313.

Medanic J. and M. Andjelic (1971). On a class of differential games without saddle-point solutions, *J. of Optimization Theory and Applications*, **Vol. 8**, pp. 413–430.

Richter, S. (1987). A homotopy algorithm for solving the optimal projection equations for fixed order dynamic compensation: existence, convergence and global optimality, In *Proc. of ACC.*

Rotea M. A. and R. K. Prasanth (1994). The ρ performance measure: A new tool for controllers design with multiple frequency domain specifications, In *Proc. of ACC*, pp. 430–435.

Rudin, W. (1966). *Real and Complex Analysis*, McGraw-Hill, New York.

Youla, D. C., H. Jabr and J. Bongiorno Jr. (1976). Modern Winener-Hopf design of optimal controllers — Part I, *IEEE Trans. on Auto. Control*, **AC-21**, pp. 3–13.