Numerical Solution of the Multiple Objective Control System Design Problem for SISO Systems

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Abstract: This paper presents a numerical solution of the multiple objective control system design problem for SISO systems. An approximation for the free transfer function in the Q-parametrization, in terms of controller complexity, is proposed. Two examples are presented to show that the proposed method has the advantage of directly yielding low-order controllers.

I Introduction

In real control systems, the controller is often required to meet different and even conflicting performance and robustness specifications. Even in the case where the system under consideration is linear time-invariant (LTI), the modelling and computational requirements of this situation are largely unresolved. Only for a few very special cases are there analytic methods for finding the exact form of the trade-offs among different specifications [1, 8]. However some progress has been made on numerical designs for an LTI controller which stabilizes a given LTI plant and meets multiple performance criteria. Most numerical methods proceed by solving approximate problems. These include approximate scalarization [14], U-parametrization [6, 5], iterative H_∞ optimization [9], and the more general and attractive convex optimization approach [2, 13, 3]. In the convex optimization approach, parametrizing all stabilizing linear controllers in terms of a stable transfer function Q allows many specifications to be formulated as convex constraints on the free design parameter Q. Then Q is approximated by simpler basis functions to reduce the computational complexity. In most cases, Q is approximated by a finite-impulse response (FIR) filter, so that convex constraints on Q become convex constraints on the filter coefficients. The convex optimization approach always finds a solution if one exists. However, the parameter space is usually very large and high order controllers generally result, which must then be judiciously reduced in order to be made feasible in practice. Until now, little work has been done on how to approximate the free design parameter Q to reduce the size of the parameter space and therefore the order of the designed controller.

In this paper, we present an approximate numerical solution of the multiple-objective control system design problem for SISO systems by using a specific re-parametrization of the free design parameter Q. We show through numerical examples that low order controllers can be explored by the proposed re-parametrization. In the following section, we review the formulation of the convex optimization program in H_∞ for the multiple objective control system design problem. In Section III, we propose a scheme to approximate the free design parameter Q. In Section IV we present two examples to illustrate the results of our computational procedure. We conclude with some remarks in the last section.

II Problem Formulation

The problem treated here concerns the multiple objective control system pictured in Fig. 1, where the signals $w_1, ..., w_m; u; y; z_1, ..., z_m$ are vector-valued functions of time or frequency. For each $k \in \underline{m} := \{1, 2, ..., m\}$, w_k denotes an exogenous input, while the vector z_k represents the output vector to be regulated. The vector signals u and y represent the plant's control input and measurement output. The transfer functions G and K represent the plant and controller respectively. It is assumed that G and K are real-rational, proper and that all weighting functions have been absorbed in the plant G.

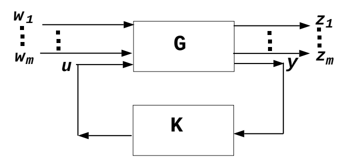

Figure 1 The multiple objective control problem

Let T_k denote the closed-loop transfer function from w_k to z_k. ***The multiple objective control problem*** is defined as follows: find all real-rational proper controllers K that internally stabilize G and minimize the maximum of the various norms (e.g., $l_1, \mathcal{H}_2, \mathcal{H}_\infty, \cdots$) of transfer functions T_k $(k \in \underline{m})$; that is, solve the following optimization problem:

$$(P^0): \gamma^0 = \min_{stabilizing\ K} \max \left\{ \|T_k(K)\|_{p_k}, k \in \underline{m} \right\}, \quad (1)$$

where, for each k, p_k may be chosen from 1, 2 or ∞.

As shown in [15, 4], the set of all controllers K that stabilize a given plant G is given parametrically as follows:

$$K(s) = (Y - MQ)(X - NQ)^{-1}, X - NQ \neq 0. \quad (2)$$

Here $Q(s)$ is the parameter, free to range over all stable real-rational transfer functions, while $N, M, X,$ and Y are fixed stable real-rational transfer functions chosen to satisfy the coprime factorization conditions

$$G = NM^{-1} \text{ and } XM - YN = I. \quad (3)$$

In terms of the Q-parametrization of the stabilizing controllers in (2), each transfer function T_k $(k \in \underline{m})$ becomes affine in Q, i.e.,

$$T_k = T_{k,1} + T_{k,2}QT_{k,3}, \ k \in \underline{m}, \quad (4)$$

where $T_{k,1}, T_{k,2}$ and $T_{k,3},\ k \in \underline{m}$, are known transfer functions, which can be obtained directly from the plant data. So the above problem (P^0) reduces to the following ***unconstrained convex optimization program*** in $\mathcal{H}_\infty$:

$$(P^1):\ \gamma^1 = \min_{Q \in \mathcal{RH}_\infty} \max\left\{\|T_k(Q)\|_{p_k}, k \in \underline{m}\right\}. \quad (5)$$

More generally, it is also shown in [13, 2, 3] that many specifications, on frequency domain response, time domain response, and various norms of the closed-loop transfer matrices, can be formulated as inequalities of the form $\psi_k(Q) \leq 0,\ k \in \underline{m}$, where $\psi_k(\cdot) : \mathcal{RH}_\infty \rightarrow \mathcal{R},\ k \in \underline{m}$, are convex (but perhaps nondifferentiable). Thus the multiple objective control system design can be viewed as a special case of the following ***unconstrained convex optimization program*** in $\mathcal{H}_\infty$:

$$(P^2):\ \gamma^2 = \min_{Q \in \mathcal{RH}_\infty} \max\{\psi_k(Q), k \in \underline{m}\}. \quad (6)$$

In addition to the many specifications, one may define a objective function $\psi_0(Q)$, where $\psi_0(\cdot) : \mathcal{RH}_\infty \rightarrow \mathcal{R}$ is convex. Then the system design problem can be formulated as a ***constrained convex optimization program***:

$$(P^3):\ \gamma^3 = \min_{Q \in \mathcal{RH}_\infty} \{\psi_0(Q) : \psi_k(Q) \leq 0, k \in \underline{m}\}. \quad (7)$$

III Finite dimensional approximations

The convex programs $(P^1), (P^2)$ and (P^3) for Q formulated in the previous section are infinite dimensional and generally cannot be solved analytically except in some special cases. Approximate solutions for Q can be found numerically by parametrizing the unknown Q. In most cases, Q is parametrized in such a way as to preserve convexity by setting

$$Q^L_{cvx}(x) = \sum_{i=1}^{L} x_i Q_i, \quad (8)$$

where the Q_i are fixed stable basis functions and $x := [x_1, x_2, ..., x_L]^T \in R^L$ is the vector of new design parameters.

With the approximation of Q by Q^L_{cvx}, $(P^1), (P^2)$ and (P^3) become finite-dimensional convex optimization programs for the unknown x in R^L. There are some powerful algorithms and software packages [13, 2, 3] designed specifically for such programs. One important property of a convex optimization program is that every local solution is actually a global solution, so there is no danger of getting "stuck" at a local minimum. Another important feature of the parametrization is that even for such straightforward choices as $Q_i(s) = \left(\frac{s-1}{s+1}\right)^{L-i}$ [13, 3], any desired Q in $\mathcal{RH}_\infty$ can be approximated arbitrarily well by some $Q^L_{cvx}(x)$ provided L is sufficiently large. Complementing this advantage, however, is the problem that adequate approximations often require large dimensions L, and correspondingly high-order approximate controllers. To circumvent this, we propose to sacrifice convexity and further parametrize the transfer function $Q(s) \in \mathcal{RH}_\infty$ as follows:

$$Q^N(x, y, s) = \frac{\sum_{i=1}^{2N+1} x_i s^{2N+1-i}}{\prod_{j=1}^{N}\left(s^2 + y_{j1}^2 s + y_{j2}^2\right)}. \quad (9)$$

Here, N is chosen to give a desired degree of accuracy for the minimal value, $x = (x_1, ..., x_{2N+1})^T \in R^{2N+1}$, $y = (y_{11}, y_{12}, ..., y_{N1}, y_{N2})^T \in R^{2N}$.

Comparing the parametrization in (9) with that in (8) reveals that Q^L_{cvx} is one special form of Q^N. Indeed, if y in Q^N is specified, i.e., the poles of Q^N are fixed in the left half plane, then the form of Q^N becomes the form of Q^L_{cvx}. Therefore any transfer function in $\mathcal{RH}_\infty$ can also be approximated arbitrarily well by the form of (9), and if the orders of Q^N and Q^L_{cvx} are equal, then Q^N should approximate Q at least as well as with Q^L_{cvx}. To see this, note that we can always choose the solution of Q^L_{cvx} in the convex program as the initial guess of Q^N in the nonlinear program by an appropriate mapping and then make further improvements by adjusting parameters of Q^N in the nonlinear program. More importantly, the form of (9) allows the direct placement of $2N$ poles (some real, some in complex-conjugate pairs) at arbitrary points in the left half plane, in contrast to the fixed poles of Q^N. This is the key to limiting controller complexity: our method usually produces controllers of lower order than those produced by other methods, especially the convex optimization method. In the next section, two sample design problems are solved to show the results of the computational procedure.

IV Computational Aspects and Examples

With the transfer function Q parametrized as in (9), the convex programs $(P^1), (P^2)$ and (P^3) are approximated by semi-infinite optimization programs. There are some specifically well-developed algorithms for solving these, e.g., [12, 11], which have been shown to be quite successful. In our computations, we use the ***Optimization Toolbox for Use with MATLAB*** [7]. To use this, we make one naive approximation. We simply approximate semi-infinite constraints and objectives by discretization, e.g., replacing a H_∞ norm constraint by a very large number of single frequency constraints log-spaced in a specified frequency range. No doubt great improvements in performance would result from the use of sophisticated methods for semi-infinite optimization programs such as those described in [12, 11].

Initial experimentation with the proposed solution of the multiple objective control system design problem for SISO systems is certainly encouraging. Even though our method is not guaranteed to find the global minimum of the objective function, computational examples show that an acceptable suboptimal solution can always be obtained with a little common sense. First, solutions from lower order problems can generally be used as starting points for higher order problems by using an appropriate mapping. Second, note that we can write $Q^N(x, y) = N_Q(x) D_Q^{-1}(y)$, where $N_Q(x)$ is convex. Based on this observation, we can improve the convergence by updating the starting points as follows: 1) get a solution (x^1, y^1), 2) fix $D_Q(y)$ by setting $y = y^1$, and solve a convex optimization program for $N_Q(x)$ to get a solution x^2, 3) take (x^2, y^1) as a new starting guess at the solution. Third, we can specify the controller structure and verify the final results based on the information about the performance limit provided by the convex optimization approach. Two examples are now given to illustrate the design procedure.

Example 1 [9]: Consider an unstable plant whose transfer function is

$$G(s) = \frac{-s+10}{s^2-0.5s+1}.$$

It is required to design a controller $K(s)$ which will stabilize

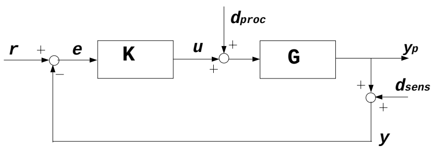

Figure 2 General closed-loop system

the plant in a negative unity feedback configuration (Fig. 2), such that the sensitivity function $T_{er} = (I+GK)^{-1}$ and the inverse additive stability margin $T_{ur} = K(I+GK)^{-1}$ satisfy

$$|T_{er}(j\omega)| \le |l_1(j\omega)|,\ \forall\omega, \tag{11}$$

and

$$|T_{ur}(j\omega)| \le |l_2(j\omega)|,\ \forall\omega, \tag{12}$$

where the bounding functions $l_i(s)$ $(i=1,2)$ are given by

$$l_1(s) = 2\left(\frac{s+0.01}{s+4.5}\right) \tag{13}$$

and

$$l_2(s) = 10\left(\frac{s+2}{s+10}\right)^2. \tag{14}$$

If we define weights W_1 and W_2 as $W_1 = l_1^{-1}$ and $W_2 = l_2^{-1}$ respectively, then the closed-loop system can be represented by Fig. 1 with $T_1 = W_1(I+GK)^{-1}$ and $T_2 = W_2K(I+GK)^{-1}$. The design specifications given in (11) and (12) will be met if and only if

$$max\{\|T_1(K)\|_\infty, \|T_2(K)\|_\infty\} \le 1. \tag{15}$$

Therefore we can formulate the following minimax optimization problem:

$$(P^4):\ \gamma^4 = \min_{stabilizing\ K} \max\{\|T_1(K)\|_\infty, \|T_2(K)\|_\infty\}. \tag{16}$$

If the solution $\gamma^4 \le 1$, then minimizing controller K meets the design specifications.

The solution of problem (P^4) proceeds by first obtaining a stable right coprime factorization of G as in (3) with

$$\begin{aligned} N &= \frac{-s+10}{s^2+3s+2},\ M = \frac{s^2-0.5s+1}{s^2+3s+2}, \\ X &= \frac{s^2+6.5s+16.5}{s^2+3s+2},\ \text{and}\ Y = \frac{-1.25s+1.25}{s^2+3s+2}. \end{aligned} \tag{17}$$

In terms of the Q-parametrization of the stabilizing controllers, the set of achievable closed-loop transfer functions T_i $(i=1,2)$ can be parametrized as

$$\{T_{i,1} + T_{i,2}Q,\ Q \in RH_\infty\}, \tag{18}$$

where $T_{i,1}$ and $T_{i,2}$ $(i=1,2)$ are the following stable transfer functions:

$$\begin{aligned} T_{11} &= W_1MX,\ T_{12} = -W_1MN,\ \text{and} \\ T_{21} &= W_2MY,\ T_{22} = -W_2MM. \end{aligned} \tag{19}$$

Thus the original problem (P^4) is equivalent to the following unconstrained convex optimization program:

$$\left(P^{4'}\right):\ \gamma^4 = \min_{Q\in RH_\infty} \max\{\|T_1(Q)\|_\infty, \|T_2(Q)\|_\infty\}. \tag{20}$$

By approximating Q using the parametric expression (9), we reduce $\left(P^{4'}\right)$ to the following unconstrained semi-infinite optimization program

$$(P^5):\ \gamma_N^5 = \min_{x,y} \max\{\|T_1(x,y,\omega)\|_\infty, \|T_2(x,y,w)\|_\infty\}. \tag{21}$$

Finally, this program can be approximated by discretization and then solved by using the "minimax" function in the ***Optimization Toolbox*** [7].

When $N \ge 2$, $\gamma_N^5 < 1$ already meets the required specifications (see Table 1). For example, approximating $Q(s)$ by

$$Q^2(s) = 9.7028\,\frac{\frac{(s^2+5.9987s+10.3731)}{(s^2+0.6651s+1.1342)}}{\frac{(s+14.3812)(s+4.8155)}{(s^2+0.5003s+1.000)}}, \tag{22}$$

directly yields the minimal realization of a 6th-order controller according to (2)

$$K(s) = 9.7028\,\frac{\frac{(s^2+3.9993s+4.0783)}{(s^2+1.9722s+0.9763)(s^2+0.3215s+0.7135)}}{\frac{(s^2+1.9514s+0.9618)(s+29.9167)(s+0.0094)}{(s^2+4.0226s+4.1561)}} \tag{23}$$

and the corresponding minimal objective value is $\gamma_2^5 = 0.9718 < 1$.

With this controller, the frequency responses of objective functions T_1 and T_2 are shown in Fig. (3), the unweighted sensitivity function $T_{er}(j\omega)$ together with its bounding function $l_1(j\omega)$ and the robustness indicator $T_{ur}(j\omega)$ together with its bounding function $l_2(j\omega)$ are depicted respectively in Fig. (4) and Fig. (5). The graphs show that the designed controller meets the required specifications given in (11) and (12), and $\max\{|T_1(j\omega)|, |T_2(j\omega)|\}$ is almost "flat". Therefore even $Q^2(s)$ gives a very reasonable solution to this example.

To quantitatively verify our results, we also approximately solved the convex optimization program $\left(P^{4'}\right)$ in (20) by parametrizing $Q(s)$ in the form of (8):

$$Q_{cvx}^{50} = \sum_{i=1}^{50} x_i \left(\frac{s-1}{s+1}\right)^{50-i}. \tag{24}$$

The objective value for the approximation to this problem is 0.9666. Increasing the number of parameters in Q_{cvx}^L does not significantly improve the solution; this gives some confidence that the solution based on (24) is nearly optimal. The controller

K obtained from Q^{50}_{cvx} is a 56th-order stable controller, which has a 51st-order minimal realization.

The controller design problem in this example was also solved in [9] by a nested iterative H_∞ optimization procedure. This is computationally very demanding and the order of the desired controller can increase very rapidly with the number of iterations. Our method produces better results even when $N = 2$. Model reduction techniques were also used in [9].

Some features of the solutions described above are tabulated in Table 1. The table shows that the numerical sequence $\{\gamma^5_N\}$ generated by our proposed method provides an excellent approximation to the nearly optimal value computed by the convex optimization method. Furthermore our proposed method leads to controllers of much lower order than those produced by both the convex optimization method and the nested iterative H_∞ optimization procedure.

Example 2 [14, 10, 5] : Consider the problem of minimizing

$$f(K) = \left\|W_1(I+GK)^{-1}\right\|_\infty \tag{25}$$

subject to the multiplicative robust stability constraint

$$g(K) = \left\|W_2GK(I+GK)^{-1}\right\|_\infty \le 1, \tag{26}$$

where

$$\begin{aligned} G &= \frac{s-2}{s-12};\ W_1 = \frac{1}{30}\frac{s+6}{s+1}; \\ W_2 &= \frac{1}{3}\frac{s+1}{s+2}\frac{(s+6)^2}{(s^2+2s+37)}. \end{aligned} \tag{27}$$

A stable coprime factorization of G as in (3), taken from [14], is

$$\begin{aligned} N &= \frac{s-2}{s+6}; M = \frac{s-12}{s+6}; \\ X &= -0.8;\ \text{and}\ Y = -1.8. \end{aligned} \tag{28}$$

In terms of the Q-parametrization in (2) and the approximation of Q by Q^N in (9), we can get an approximate solution of this problem by solving the following constrained semi-infinite optimization program:

$$\left(P^6\right):\ \gamma^6_N = \min_{x,y}\{f(x,y,\omega) : g(x,y,\omega) - 1 \le 0\}. \tag{29}$$

For $N = 1$, solving $\left(P^6\right)$ directly gives

$$Q_1(s) = 1.1415\frac{s^2 - 0.8292s + 14.6680}{s^2 + 4.8751s + 16.7021}, \tag{30}$$

and the corresponding minimal realization of a third order controller is

$$K(s) = -1.5151\frac{(s-0.1837)(s^2+1.86005s+38.0154)}{(s+0.8968)(s^2+1.9209s+26.8114)}. \tag{31}$$

With this controller

$$f(K) = 0.1916\ \text{ and }\ g(K) = 1.0000. \tag{32}$$

This numerical example was also solved in [14] by approximate scalarization, in [10] by convex optimization, and in [5] by U-parametrization methods. Table 2 shows the results obtained by these three methods and our proposed method. It seems that the sequence obtained by our proposed method $\{\gamma^6_N\}$ converges and our proposed method offers the best result in terms of the smallest objective value and the lowest controller order.

V Concluding remarks

An approximate numerical solution is proposed for the multiple objective design of robust SISO systems. The key is a nonconvex re-parametrization of the transfer function Q in the Q-parametrization in terms of controller complexity. Two examples illustrate the effectiveness of the proposed technique. They seem to indicate that the approximate solution generated by the ***Optimization Toolbox*** is very close to at least a local minimum. However, the approximate solution can be further refined, if so desired, by using sophisticated semi-infinite optimization algorithms. In comparison to other methods, especially the convex optimization method, the design procedure presented here has the advantage of directly yielding low-order controllers.

Method	The minimal objective value	The controller order (minimal realization)
Convex optimization	0.9666	51
The iterative H_∞ [9]	0.9729	7(after model reduction)
The proposed $(N = 1)$	1.1756 (>1, not desirable)	Not applicable
The proposed $(N = 2)$	0.9718	6
The proposed $(N = 3)$	0.9703	8
The proposed $(N = 4)$	0.9683	12

Table 1 Solutions of Example 1

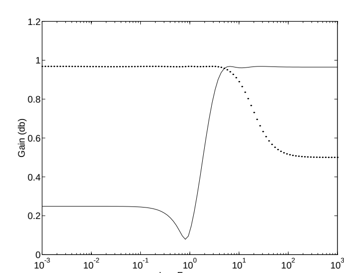

Figure 3 Weighted objective functions: $|T_1(j\omega)|$(dotted line) and $|T_2(j\omega)|$(solid line)

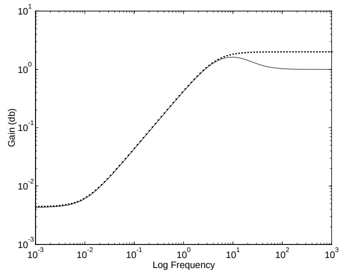

Figure 4 Sensitivity function T_{er}(solid line) and its bounding function l_1(dotted line)

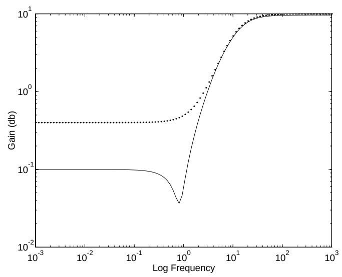

Figure 5 Robustness function T_{ur}(solid line) and its bounding function l_2(dotted line)

Method	$f(K)$	$g(K)$	The controller order (minimal realization)
Convex optimization [10] (using 20-tap Q)	0.1864	1.0000	20
Convex optimization [10]	0.1995	0.9929	3 (after model reduction)
U-parametrization [5]	0.2943	0.9361	3
Approximate scalarization [14]	0.7998	0.9999	3
The proposed $(N = 1)$	0.1916	1.0000	3
The proposed $(N = 2)$	0.1779	1.0000	5
The proposed $(N = 3)$	0.1776	1.0000	7

Table 2 Solutions of example 2

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