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Abstract

Signal transmission in terms of the time and frequency dimension have been areas of research in coding and signal processing domains for many years. Results from these research areas have lead to mobile system designs which have good performance with acceptable complexity. However, new demands in terms of information transmission and reliability cannot be achieved in the time and frequency domain alone, without a large increase in complexity, or resources.

For this reason, the spatial dimension has become an area of interest. Mobile communication systems incorporating multiple transmit and receive antennas have become an area of research interest, and such channels have been termed spatial multiple input / multiple output (MIMO) systems. This lab investigates several major aspects of the spatial MIMO channel model:

- Error performance improvement through spatial diversity
- Energy normalization of multiple transmit/receive antenna systems
- Modeling of the Rayleigh spatial MIMO channel
- Comparison of orthogonal diversity codes to beamforming systems

Based on the knowledge gained, a subsequent lab is offered which focuses on the application of the concepts of coding theory to the spatial MIMO channel scenario.
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### List of symbols

#### Operations
- $\cdot$ : vector notation
- $\cdot^c$ : complex conjugate operation
- $\cdot^H$ : Hermitian conjugate transpose operation
- $\cdot^{-1}$ : matrix inverse operation
- $|\cdot|$ : absolute value operation
- $E_x \{\cdot\}$ : expectation with respect to $x$
- $x \mod \Pi$ : $x$ modulo $\Pi$

#### Roman Symbols
- $\mathcal{A}$ : signal constellation set
- $|\mathcal{A}|$ : cardinality of the signal constellation
- $\mathbf{A}(x, \tilde{x})$ : Hermitian codeword difference matrix between $x$ and $\tilde{x}$
- $a$ : real part of the Gaussian prime $\Pi$
- $b$ : imaginary part of the Gaussian prime $\Pi$
- $\mathbb{C}$ : complex field
- $d_x$ : distance with respect to the metric $x$
- $\text{det}(\cdot)$ : determinant operation
- $E_b/N_0$ : bit energy to noise energy ratio
- $E_s/N_0$ : symbol energy to noise energy ratio
- $\mathbb{F}$ : finite field
- $f_x(\cdot)$ : norm operation with respect to the metric $x$
- $G_{\Pi}(p)$ : Gaussian integer number field for $\Pi$ over $p$.
- $\mathbf{H}$ : channel coefficient matrix
- $h$ : complex channel coefficient
- $\mathbf{I}$ : identity matrix
- $\mathcal{M}$ : set of matrices
- $\mathbf{M}$ : arbitrary matrix
- $m_{ij}$ : matrix element
- $N_f$ : number of frequency dimensions
- $N_{xx}$ : number of receive antennas
- $N_s$ : number of spatial dimensions
- $N_t$ : number of timeslots
- $N_{tx}$ : number of transmit antennas
- $P_x$ : error probability with respect to $x$
- $p$ : prime number
- $p(x)$ : primitive polynomial over $x$
- $R_{\text{sym}}$ : symbol rate of the code
$R_t$ ................................................................. temporal rate of the code
$r$ ................................................................. received symbol
$S$ ................................................................. space time code
$s$ ................................................................. complex symbol
$U$ ................................................................. unitary matrix
$U(x_1x_2)$ ........................................ Uniform distribution with bounds $x_1$ and $x_2$.
$W$ ................................................................. antenna coefficient
$x$ ................................................................. real part of a Gauss integer
$y$ ................................................................. imaginary part of a Gauss integer
$Z$ ................................................................. integer field

**Greek Symbols**

$\alpha$ ........................................................ primitive element
$\zeta$ ........................................................ Gaussian integer
$\lambda$ ........................................................ electrical wavelength
$\theta$ ........................................................ beam angle
$\rho(x)$ ....................................................... Rayleigh distribution with parameter $x$
$\eta$ ........................................................ noise term
$\lambda_i$ ....................................................... eigen value $i$
$\mu_x$ ....................................................... mean with respect to $x$
$\Pi$ ........................................................ Gaussian prime number
$\sigma^2_x$ ................................................ variance with respect to $x$
$\varphi$ .................................................... uniformly distributed random angle between $\pm \pi$
# List of abbreviations

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<th>Description</th>
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<td>BER</td>
<td>Bit Error Rate</td>
</tr>
<tr>
<td>BPSK</td>
<td>Binary Phase Shift Keying</td>
</tr>
<tr>
<td>CER</td>
<td>Codeword Error Rate</td>
</tr>
<tr>
<td>DOA</td>
<td>Direction of Arrival</td>
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<td>Demod.</td>
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<td>Equ.</td>
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<td>QPSK</td>
<td>Quadrature Phase Shift Keying</td>
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<td>MIMO</td>
<td>Multiple Input / Multiple Output</td>
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<td>ML</td>
<td>Maximum likelihood</td>
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<tr>
<td>Mod.</td>
<td>Modulation</td>
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<td>MPSK</td>
<td>M-Phase Shift Keying</td>
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<td>MRC</td>
<td>Maximum Ratio Combing</td>
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<td>MRD</td>
<td>Maximum rank distance</td>
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<td>Orth.</td>
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<td>Sep.</td>
<td>Separation</td>
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<tr>
<td>SER</td>
<td>Symbol Error Rate</td>
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<td>SISO</td>
<td>Single Input / Single Output</td>
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<td>SNR</td>
<td>Signal to Noise Ratio</td>
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<td>STC</td>
<td>Space Time Code</td>
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Chapter 1

Signal transmission over time, frequency and space

In this chapter we introduce the basic single input / single output (SISO) channel model and describe it in terms of the time, space and frequency dimensions. We then extend this channel model to give the multiple input / multiple output (MIMO) channel model. In particular we consider the spatial MIMO channel, and compare some of its similarities and differences with that of the SISO channel.

1.1 SISO channel model

We begin this section with an example to illustrate the terms used to describe the SISO channel

Example 1 (SISO channel model) Consider the channel model shown in figure 1.1. We transmit a vector codeword of BPSK transmission symbols $\mathbf{s} \in \{\pm 1\}$ of length $N_t = 5$ from a single transmit antenna $N_{tx} = 1$. There are no multipath components, and each channel coefficient $h_i$ is an independently, identically distributed random variable. The transmitted signal is captured by a single receive antenna $N_{rx} = 1$. Furthermore, we assume a narrow band system which occupies one transmission frequency and experiences flat fading. In other words, the frequency occupied by the transmitted symbol $s_i$ is affected by only one attenuation term in the frequency dimension. A graphical representation of the system is shown in figure 1.2. The received signal for time instant $t$ is described mathematically as:

$$r(t) = h(t) \cdot s(t) + \eta(t). \quad (1.1)$$
1.1.1 Temporal transmission dimension, $N_t$

In example 1, the temporal description of the codeword consists of $N_t$ individual symbols which are transmitted in separate timeslots. This corresponds to the x-axis in figure 1.2. Based on the channel assumptions, each timeslot in the temporal dimension can be considered as an orthogonal basis set. We may write the transmission vector in terms of the temporal basis vectors

$$\vec{s}_{temp} = s_1 \cdot (1,0,0,0,0) + s_2 \cdot (0,1,0,0,0) + s_3 \cdot (0,0,1,0,0) + s_4 \cdot (0,0,0,1,0) + s_5 \cdot (0,0,0,0,1).$$

1.1.2 Frequency transmission dimension, $N_f$

The frequency dimension in example 1 is the y-axis of figure 1.2. Based on the assumptions in the example, the codeword vector has only one value in the frequency dimension which is the same for all symbols. As for the temporal dimension, we may write each frequency dimension as an orthogonal basis vector. The representation of the codeword vector in the frequency dimension is as follows

$$\vec{s}_{freq} = s_1 \cdot (0,1,0,0,0) + s_2 \cdot (0,1,0,0,0) + s_3 \cdot (0,1,0,0,0) + s_4 \cdot (0,1,0,0,0) + s_5 \cdot (0,1,0,0,0).$$

1.1.3 SISO spatial transmission dimension, $N_s = 1$

In example 1, we have only one transmit and receive antenna, therefore it is clear that there is only one spatial dimension used. The transmitted codeword in the spatial dimension can also be described
1.2 Spatial MIMO channel model

As in section 1.1, we begin this section with an example.

![Multiple input / multiple output system](image)

**Example 2 (Spatial MIMO channel model)** Consider the channel model shown in figure 1.3. We transmit a matrix codeword of BPSK transmission symbols $s \in \{\pm1\}$ with length $N_t = 5$ from $N_{tx} = 5$ transmit antennas. The signal is recovered at the receiver by $N_{rx} = 5$ receive antennas. Furthermore, we assume a narrow band system which occupies one transmission frequency, and experiences flat fading as in example 1. The received signal for a particular timeslot $t$, and receive antenna $j$ is given as:

$$r_j(t) = \sum_{l=1}^{N_{tx}} h_{lj}(t) \cdot s_l(t) + \eta_j(t).$$

(1.2)

In the temporal and frequency dimensions, it is necessary to describe the transmission symbol $s_{lj}$ using two indices, where the second index $l$ corresponds to the transmit antenna index. With this exception, the temporal and frequency dimensions at the transmitter have the same considerations as in section 1.1. We next focus on the spatial dimension, and its associated physical properties.

1.2.1 MIMO spatial transmission dimension, $N_s = \min\{N_{tx}, N_{rx}\}$

The spatial dimension is defined by the minimum number of transmit or receive antennas. In example 2, we use $N_s = \min\{N_{tx}, N_{rx}\} = \min\{5, 5\} = 5$ spatial dimensions. To understand the rational of this statement, we consider an analogy using the time and frequency dimensions. The number of
timeslots / frequencies of the transmitted signal is identical to the number of timeslots / frequencies at the receiver. The same argument is made for the spatial dimension where the number of antennas correspond to the number of spatial dimensions.

To see this relationship mathematically, we consider a system of equations given by the received signals per antenna. Based on equation 1.2 without noise, we may write the received vector for time slot $i$ as

$$\mathbf{H}_i \mathbf{s}_i = \mathbf{r}_i.$$  

(1.3)

If we assume that the channel coefficients are known at the receiver, then we have a linear equation with 5 unknowns $s_{i,l}$, $1 \leq l \leq 5$ and 5 equations. If we further assume that the matrix of channel coefficients $\mathbf{H}_i$ has an inverse, then we may solve the equation for $\mathbf{s}_i$ as follows

$$\mathbf{s}_i = \mathbf{H}_i^{-1} \cdot \mathbf{r}_i.$$  

(1.4)

From this perspective, it is clear that if $N_{rx} < 5$ then the system cannot be solved. If $N_{rx} > 5$, then the system is over determined. Furthermore, if the matrix $\mathbf{H}_i$ does not have an inverse, which is equivalent to saying that it does not have full rank (i.e. 5 linearly independent rows or columns), then
this system also cannot be solved. The solution presented in equation 1.4 is called zero forcing in the signal processing literature, and may be applied to any scenario of this form.

When we introduce the noise vector $\hat{\eta}$ into the channel model, then we cannot exactly solve for the transmitted signal. Instead, we can get an estimate of which signal was transmitted based on a particular distance measure.
Chapter 2
Spatial channel considerations

In chapter 1 we presented a model for the spatial MIMO channel. In this chapter we present the concept of channel diversity, and introduce some necessary channel assumptions.

2.1 Channel diversity

An important concept for the description of a channel is the number of statistically independent fading coefficients $h$ (p. 777 [Pro95]). This number is referred to as the channel diversity. It is a property of the system, and is independent of the signal constellation or channel code selected. The challenge with respect to signal transmission, is to find a way of exploiting this diversity. This involves the construction of channel codes, and efficient decoders at the receiver. The following definition is based on the eigen decomposition of a matrix, which is addressed on most algebra textbooks.

**Definition 1 (Channel diversity)** The diversity of the channel is defined by the number of statistically independent channel coefficients in the space, time and frequency dimensions. This is given by the number of non-zero eigenvalues (rank) of the channel correlation matrix

$$
E_h \{H \cdot H^H\} = \begin{bmatrix}
U & \\
\lambda_1 & 0 & \\
0 & \ddots & \\
& & \lambda_N
\end{bmatrix}
\begin{bmatrix}
U & \\
0 & \\
& \ddots
\end{bmatrix}^H,
$$

where $(\cdot)^H$ is the Hermitian complex transpose operator, $H$ is the channel coefficient matrix, $U$ is a unitary matrix where the columns are the eigenvectors of $E_h \{H \cdot H^H\}$, $\lambda_i$ is an eigenvalue and $E_h \{\cdot\}$ is the expectation operation with respect to the random variable $h$.

Diversity can only be exploited when the same information is transmitted over the different diversity dimensions. In other words, there must be a type of repetition code in the transmission of the information. Therefore designing a transmission system for diversity means that the information rate must be sacrificed. We further explain this concept using the two examples from sections 1.1 and 1.2.

In example 1, there are 5 different channel coefficients in the temporal dimension $h_{SISO}^{SISO} = [h_1, h_2, \ldots, h_5]$. It is assumed that the channel coefficients are all statistically independent which gives 5 non-zero eigenvalues, therefore this channel has a diversity of 5.

$$
E_h \{h \cdot h^H\} = \begin{bmatrix}
1 & 0 & \\
\ddots & \ddots & \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 & \\
0 & \ddots & \\
0 & 0 & \lambda_5
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \\
\ddots & \ddots & \\
0 & 0 & 1
\end{bmatrix}.
$$
To understand the concept of diversity, we assume a simple channel model where a channel coefficient is erased ($|h| = 0$) with probability $P_e$. We then transmit 5 information symbols over the 5 timeslots, giving an information rate of one. Each information symbol has a probability of $P_h$ of being erased by the channel.

Next, we consider a repetition code where a single information symbol is transmitted over all 5 timeslots giving an information rate of $1/5$. The probability that the information is completely erased by the channel is $(P_e)^5$. We have sacrificed the information transmission rate in favor of more reliable transmission by exploiting the diversity of the channel.

For example 2, the analysis is a bit more complex. We must consider a correlation matrix for every receive antenna for the 5 transmission paths. For receiver antenna $j$ we have

$$E_{h} \left\{ \vec{h}_j^H \cdot \vec{h}_j \right\}^{MIMO} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_M^j \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}^{(H)}.$$

Therefore each receive antenna has 5 eigenvalues and gives a total channel diversity of 25 for the system. In chapter 3 we consider ways of signal transmission to exploit diversity in the spatial dimension.

The effect of the channel diversity on the probability of an error can be seen in the slope of the error performance curve. For example, in figure 2.1 we consider the transmission over a SIMO system with two receive antennas $N_{rx} = 2$, and a SISO system.

![Figure 2.1: Effect of diversity on the error performance.](image)

Each path from transmitter to receiver is statistically independent, or in other words, the channel has a diversity of two. We calculate the slope of both error performance curves for the points $E_b/N_0$ and $\tilde{E}_b/N_0$ using the following formula:
2.2 CHANNEL ASSUMPTIONS

This gives a slope of -2.0 / dB and -1 / dB for the SIMO and the SISO curves respectively. The slope difference arises from the channel diversity. The higher the channel diversity, the steeper is the slope of the error performance curves.

2.2 Channel assumptions

For the purpose of system evaluation, we consider a particular channel model and several simplifying assumptions which are summarized below.

Each receive antenna has an independent complex zero mean additive white Gaussian noise source defined by

\[
\eta = \eta_{Re} + j \cdot \eta_{Im},
\]

where \(\eta_{Re}\) and \(\eta_{Im}\) are the real and imaginary components of the noise terms respectively. The Gaussian noise distribution for both \(\eta_{Re}\) and \(\eta_{Im}\) is given by \(N(\mu, \sigma^2)\) with mean \(\mu = 0\) and variance \(\sigma^2 = \sigma^2_\eta\). We assume that the real and imaginary part of the noise are uncorrelated and have the identical distribution, i.e. \(\mathbb{E}\{\eta_{Re} \cdot \eta_{Im}\} = 0\) and \(\mathbb{E}\{|\eta_X|^2\} = \sigma^2_{\eta X} = \sigma^2_\eta\) where \(X\) represents either \(Re\) or \(Im\).

The energy of the noise term is given by the expectation value:

\[
\mathbb{E}\{|\eta|^2\} = \mathbb{E}\{|\eta_{Re}|^2\} + \mathbb{E}\{|\eta_{Im}|^2\} + \mathbb{E}\{\eta_{Re}^* \cdot \eta_{Im}\} + \mathbb{E}\{\eta_{Re}^* \cdot \eta_{Im}^*\} = 2 \cdot \sigma^2_\eta.
\]

The channel coefficients are multiplicative complex random variables which are statistically independent

\[
h = h_{Re} + jh_{Im} = |h| \cdot \exp\{j \cdot \varphi\}.
\]

\(h_{Re}\) and \(h_{Im}\) have a Gaussian distribution \(N(0, 1/2)\). \(|h| = \sqrt{|h_{Re}|^2 + |h_{Im}|^2}\) is a \((1/2)\) Rayleigh distributed random variable with mean \(1/2 \cdot \sqrt{\pi/2}\) and variance \((4 - \pi)/8\). \(\varphi = \tan^{-1}(h_{Im}/h_{Re})\) has a uniform distribution \(U(-\pi, \pi)\) with maximum and minimum bounds of \(-\pi\) and \(-\pi\) respectively. The expected value of the channel energy with the above parameters is \(\mathbb{E}\{|h|^2\} = 1\) [Pap84]. We next make several assumptions used to describe the channel.

Assumption 1 (Quasi-static channel) For one codeword period of \(N_t\) all channel coefficients remain constant.

\[h_{ij}(t) = h_{ij}(t + 1) = \cdots = h_{ij}(t + N_t - 1)\]

This assumption removes the dependence of the codeword on the time dimension statistics. Such an assumption is valid in an environment where the mobile station has a low velocity. The exact bounds for such an assumption depend on specific system parameters, which will assumed to be satisfied in the following.

Assumption 2 (Optimum interleaved channel coefficients) The correlation between successive channel coefficients of a codeword period is zero.

\[\mathbb{E}\{h_{ij}(t), h_{ij}(t + N_t)\} = 0 \quad \forall \; i, j\]
Optimum interleaving destroys the correlation among successive codewords, by reordering the codewords within the interleaver period. As a result, the memory of the channel is destroyed which simplifies the construction of channel codes, while sacrificing the channel capacity which can be achieved. The interleaver introduces a corresponding delay at the decoder. Strictly speaking, to destroy codeword correlation completely requires an interleaver of infinite period and therefore an infinite delay. Practically this condition can be achieved with an acceptable delay, based on the system conditions.

Assumption 3 (Statistical independence) The set of \( N_{tx} \) quasi-static optimum interleaved channel coefficients for the \( j \)-th receive antenna \( \tilde{h}_j = [h_{1j}, \ldots, h_{N_{tx}j}] \) \( \forall 1 \leq j \leq N_r \) satisfy

\[
\mathbb{E} \{ [h_{1j}, \ldots, h_{N_{tx}j}]^H, [h_{1j}, \ldots, h_{N_{tx}j}] \} = \begin{bmatrix}
\mathbb{E} \{ h_{1j}, h_{1j}^H \} & 0 \\
0 & \ddots \\
& & \mathbb{E} \{ h_{N_{tx}j}, h_{N_{tx}j}^H \}
\end{bmatrix}
\]

Statistical independence of the antennas is comparable to orthogonal transmission paths. This condition is satisfied when the electrical distance among the antennas is large, and when there are many scatterers in the environment.

### 2.3 Energy normalization for MIMO channels

A key factor in comparing different codes and channels is the normalization of the energy. There are several different definitions that can be considered, most of which are based on the ratio of signal energy (from the transmitter) and the noise energy (from the environment) for one time period. The two most common definitions are the bit energy to noise ratio, and the symbol energy to noise ratio per timeslot. Equation 2.3 gives an relation for the \( SNR \) of an AWGN channel where the variables are scalars. \( s \) and \( h \) are the instantaneous signal and channel values respectively, while \( \sigma^2_n \) is the variance of the AWGN noise term. \( E_s/N_0 \) and \( E_b/N_0 \) are the signal energy and bit energy to noise ratio respectively. \( R_t \) is the temporal rate of the code considered in terms of information symbols per time slots \( N_t \). When no coding is used, this is set to one. \( |\mathcal{A}| \) is the cardinality of the signal constellation. The function \( \xi(x) \) is the energy as a function of the variable \( x \), while \( \xi_s \) is the average energy of the variable \( x \).

\[
SNR(h, s, \eta) = \frac{\text{Average signal information energy}}{\text{Average noise energy}} = \frac{\xi(s, h)}{\xi(\eta)}
\]

\[
SNR = \frac{\xi_{s,h}}{\xi_\eta} = \frac{\mathbb{E}_{h,s} \{ |h|^2 \}}{\mathbb{E}_{\eta} \{ |\eta|^2 \}} = \frac{E_s}{N_0} \cdot R_t = \frac{E_b}{N_0} \cdot \log_2 |\mathcal{A}| \cdot R_t
\]

The transmission of information over a channel depends on random processes with respect to the noise, symbol and channel terms. For the following equations, we consider the channel model presented in section 2.2. We assume no correlation among the noise, signal, and channel. The cross correlation of each of these random processes are all zero. For the energy of each random process (autocorrelation, i.e. when \( i = j \)) we require

\[
\xi_\eta = \mathbb{E} \{ \eta_i \cdot \eta^*_i \} = 2 \cdot \sigma^2_n, \quad \xi_s = \mathbb{E} \{ s_i^H \cdot s_i \} = \mathbb{E} \{ |s|^2 \} = 1, \quad \xi_x = \mathbb{E} \{ |x|^2 \} = 1 \forall i.
\]

\[
\xi_{\eta} = \mathbb{E} \{ \tilde{h}_i \cdot \tilde{h}_i^* \} = \mathbb{E} \{ |h|^2 \} = 1 \forall i.
\]

\[1\]In channel coding theory, the rate is defined as the number of bits per channel use, and is given as \( R = \frac{k}{n} \cdot \log_2 |\mathcal{A}| \). \( k \) is the number of information symbols, and \( n \) is the number of channel uses.
By taking the expectation value of the SNR with respect to $h$, and $s$, we get an expression that only depends on the noise variance $\sigma^2_\eta$. Under the above conditions, we ensure that the expectation value of the numerator of equation 2.3 is equal to one. This gives the noise variance as a function of the $E_b/N_0$ only.

To find the energy $\xi_x$ of a vector of uncorrelated random variables, we take the sum of the auto-correlation terms of each vector element.

$$\xi_x = \sum_{i=1}^l \mathbb{E}\{x_i^H \cdot x_i\}$$

In the following sections, we consider the calculation of the noise variance of different MIMO scenarios as a function of the $E_b/N_0$ with a normalization which satisfies the energy conditions given in equations 2.3. The channel model is the same as the one given in figure 1.3 and is modified based on the number of transmit and receive antennas. We consider uncoded transmission, i.e. $R_s = 1$ in the following.

### 2.3.1 SISO energy normalization

The received signal for a single timeslot of the SISO channel model i.e. $N_{tx} = 1$, $N_{rx} = 1$, is given by the following equation:

$$r = h \cdot s + \eta.$$  

We next derive an expression for the noise variance in which the total signal energy is normalized to unity. This is done in two steps as given in [Pro95]. First $h$ is considered as a constant, and we find the expectation $\mathbb{E}_s\{|h \cdot s|^2\}$. In the second step, we take the expectation of this result with respect to $h$. Based on the channel assumptions and equation 2.3, we get

$$\sigma^2_\eta = \frac{1}{2} \left( \frac{E_b}{N_0} \cdot \log_2 |A| \cdot R_s \right)^{-1}. \tag{2.5}$$

### 2.3.2 MISO energy normalization

From figure 1.3, we consider the case where we have $N_{tx}$ transmit and $N_{rx} = 1$ receive antennas. The received signal for one timeslot is

$$r = \bar{h} \cdot \bar{s}^T + \eta = \sum_{l=1}^{N_{tx}} h_l \cdot s_l + \eta.$$  

Under the condition that the average transmit energy per timeslot is normalized to unity, and that the average signal energy $\mathbb{E}_s\{|s|^2\} = 1$ the noise variance is given by

$$\sigma^2_\eta = \frac{N_{tx}}{2} \left( \frac{E_b}{N_0} \cdot \log_2 |A| \cdot R_s \right)^{-1}, \tag{2.6}$$

where we have again used the technique given in [Pro95]. Conceptually, we have transmitted with a signal energy of one over all antennas and increased the noise energy appropriately since the important value is the signal to noise ratio. This result shows that the noise variance is dependent on the number of transmit antennas and increases by a factor of $N_{tx}$. 
2.3.3 SIMO energy normalization

In figure 1.3, the received signal for one timeslot is considered when we have $N_{tx} = 1$ and $N_{rx}$ transmit and receive antennas respectively. Therefore the same signal $s$ arrives at all receive antennas, weighted by different channel coefficients.

$$r = \sum_{j=1}^{N_{rx}} r_j = \sum_{j=1}^{N_{rx}} h_j \cdot s + \sum_{j=1}^{N_{rx}} \eta_j \cdot s$$

Using the assumptions for the system, and the analysis technique for the noise variance found in [Pro95], we find the noise variance to be

$$\sigma_n^2 = \frac{1}{2} \left( \frac{E_b}{N_0} \cdot \log_2 |A| \cdot R_e \right)^{-1} \quad (2.7)$$

Here it is interesting to note that the noise variance is the same for the SISO and the SIMO channel.

2.4 Signal recovery through maximum ratio combining (MRC)

Maximum ratio combining is an equalization method which is optimal for independent noise samples (white noise) (p. 779 [Pro95]), and is given in figure 2.2. Here, we define $SNR_{MRC}$ which is the signal to noise ratio taken after the MRC equalization.

$$SNR_{MRC}$$

Figure 2.2: Equalization using maximum ratio combining (MRC).

The receiver makes an estimate of the channel coefficient $h_j$, $1 \leq j \leq N_{rx}$ and then multiplies the received signal $r_j$ by the conjugate of the estimate $h_j^*$. This has the effect of removing the complex component of the channel coefficient, and recovering the signal energy. Equation 2.8 shows this type of decoding for receive antenna $j$:

$$r_j \cdot h_j^* = |h_j|^2 \cdot s + h_j^* \eta_j \cdot s$$

If we consider the received signal before the demodulator of figure 2.2, we get:

$$r = \sum_{j=1}^{N_{rx}} r_j \cdot h_j^* = \sum_{j=1}^{N_{rx}} |h_j|^2 \cdot s + \sum_{j=1}^{N_{rx}} h_j^* \eta_j \cdot s$$
2.5 Maximum likelihood (ML) demodulation

It is important to note that each received antenna signal can be separately equalized, independent of all other antennas. In other words, we may perform symbol wise MRC on each transmission path. A similar type of MRC equalization cannot be done for the MIMO system. The possibility of performing MRC equalization separately on each transmission path as in equation 2.8 is impossible due to the signal superposition at the receiver from the different transmit antennas. A type of MRC decoding may be performed where the equalization coefficient is defined as the sum of all the channel coefficients, i.e. \( h_j = \sum_{i=1}^{N_{tx}} h_{ij} \) for the case of \( N_{tx} \) transmit antennas. The received estimate over all antennas is

\[
r = \sum_{j=1}^{N_{tx}} r_j \cdot (\hat{h}_j)^* = \sum_{j=1}^{N_{tx}} \left| \hat{h}_j \right|^2 \cdot s + \sum_{j=1}^{N_{tx}} \left| \hat{h}_j \right|^2 \cdot \eta_j.
\]

For the spatial MIMO channel, \( \left| \hat{h}_j \right|^2 \) is the energy of the incoherent sum of the spatial channel coefficients. The SIMO channel was able to recover the coherent sum of the energy of the spatial channel coefficients. From the triangle equality, it is clear that

\[
\left| \hat{h} \right|^2 = \left| \sum_{i=1}^{N_{tx}} h_i \right|^2 \leq \sum_{i=1}^{N_{tx}} |h_i|^2.
\]

This superposition property of the spatial channel is a major disadvantage which greatly complicates the signal recovery at the receiver. Based on the results of section 2.3, we may write the signal to noise ratio after MRC equalization as:

\[
SNR_{MRC} = N_{rx} \cdot SNR = \frac{N_{rx}}{2\sigma^2_n} \cdot \frac{E_b}{N_0} \cdot \log_2 |\mathcal{A}| \cdot R_t.
\]

The gain in \( SNR_{MRC} \) by a factor of \( N_{rx} \) is often termed the antenna gain.

2.5 Maximum likelihood (ML) demodulation

Maximum likelihood demodulation delivers the best possible results of any signal recovery method, but often has a prohibitively high complexity (p. 9 [Bos99], [Pro95]). Figure 2.3 shows the principle behind ML demodulation for an \( 8 - \text{PSK} \) signal constellation. One can imagine that the received symbol is a point in a signal space, and that each signal point occupies a coordinate in this space. Based on some distance metric, \( d_x(s_i, r) \), such as Euclidean distance, the separation between the received point \( r \) and the point \( s_i \) is calculated for all signal points. This results in \( |\mathcal{A}| \) different comparisons for the signal constellation \( \mathcal{A} \). The signal point with the smallest distance is taken as the most probable, and is given as the demodulation result.

To illustrate this concept in more detail, we consider the SISO channel with AWGN noise and channel coefficient \( h \). BPSK symbols are transmitted. The signal estimate at the receiver is \( \hat{s} \).

\[
\hat{s} = \min_{s_i} \{|r - h s_i|\}
\]

\[
= \min \left\{ \left| \begin{bmatrix} \hat{r} \\ \hat{r} \end{bmatrix} - \begin{bmatrix} h s_1 \\ h s_{-1} \end{bmatrix} \right| \right\}
\]

\[
= \min \left\{ \left| \begin{bmatrix} h s_1 + \eta \\ h s_{1} + \eta \end{bmatrix} - \begin{bmatrix} h s_1 \\ h s_{-1} \end{bmatrix} \right| \right\}
\]

\[
= \min \left\{ \left| \eta \right| \right\}
\]

\[
= \min \left\{ \left| h(s_1 - s_{-1}) + \eta \right| \right\}
\]
Here, we note that the receiver must know the channel coefficient $h$ in order to successfully demodulate the signal. The error performance curve for this scenario is shown in figure 2.4. This curve is also used as a reference in the exercises.

![Figure 2.3: ML demodulation of the point $r$ for the distance metric $d_x$ to all constellation points $s \in A$.](image)

2.6 Comparison of SISO and spatial MIMO channels

At this point, we have introduced the SISO and spatial MIMO channels. We will now emphasize some of the similarities and differences of these two systems, and mention their significance in terms of system design.

Firstly, the temporal / frequency dimensions remain orthogonal during transmission based on our assumptions as shown in the examples 1 and 2. In other words, there is no interference from symbols.
transmitted in other time or frequency dimensions as shown in equation 1.1. This property allows MRC equalization at the receiver for each transmitted signal.

In contrast, the spatial channel consists of a linear superposition of all of the signals transmitted from the different antennas at a particular time instant as described in equation 1.2. It is therefore not possible to recover an individual signal, but a system of equations must be solved as was described in section 1.2.1. MRC equalization is not possible in such a case.

The second major difference between the two systems lies in their dependence on the channel coefficients. In the SISO system, the orthogonality of the different dimensions in time and frequency are independent of the values of the channel coefficients $h$. We will always be able to solve for individual transmission symbols. For the spatial MIMO channel, the solution of equation 1.4 depends on the values of the channel coefficient matrix $H$. If the determinant of this matrix is zero, then no unique solution exists.

We make an important observation in that there is a connection between the temporal and frequency dimensions. Using the Fourier transform [PM92], any temporal signal can be described uniquely as a frequency signal and vice versa. An increase in the temporal period of a signal, has a corresponding decrease of the bandwidth of the frequency representation. From this we can see that the time and frequency dimension are different descriptions of the same signal.

The spatial dimension is completely independent of time and frequency. Relativity theory excluded, there is no transform known which can link all three dimensions. This has significant implications with respect to code design, error performance, and channel capacity.
2.7 Problems

1. You are given a SISO system where successive pairs of channel coefficients are correlated, and non successive pairs are uncorrelated:

\[ \tilde{h} = [(h_1, h_1), (h_2, h_2), (h_3, h_3)] \quad \mathbb{E} \{h_i \cdot h_i\} = 1 \quad \mathbb{E} \{h_i \cdot h_j\} = 0 \quad \forall i, j. \]

a) Find the correlation matrix of the channel \( \tilde{h} \).

b) Find the channel diversity according to definition 1.

c) Suggest a code which would exploit the diversity of this channel, while giving a rate of 1/3 i.e. one information symbol per 3 transmit symbols.

2. We consider a binary erasure channel given in figure 2.5(a). \( P_e \) is the probability that a transmitted signal is erased for each transmission instant. The following three transmission strategies are considered for the repetition codes \( C_A, C_B \) and uncoded transmission given by \( C \):

\[ \begin{align*}
\ell_1 & \rightarrow c_A(\ell_1) = (\ell_1, \ell_1, \ell_1) \\
\ell_1 & \rightarrow c_B(\ell_1) = (\ell_1, \ell_1) \\
\ell_1 & \rightarrow c(\ell_1) = (\ell_1), \quad \ell_1 \in \{0, 1\}
\end{align*} \]

The receiver can tell when a symbol is erased, and throws this symbol away when decoding. Consider the reception of a codeword from the code \( c_B \):

\[ \bar{r} = (\ell_1, \ell_1) + (0, e) = (\ell_1, e). \]

The decoder identifies the second symbol as an erasure, and makes a decision based only on the first symbol. Therefore, a decoding failure only occurs when all symbols are erased.

![Figure 2.5: Erasure channel and error plot](image)

(a) Erasure channel  
(b) Axis for error plot

Calculate the probability of a codeword error \( P_{cw} \) (Hint: see section 2.1). Plot \( \log(P_{cw}) \) as a function of \( \log(1/P_e) \) as shown in figure 2.5(b). Consider the error probability range \( 10^{-3} \leq P_e \leq 10^0 \). How does the diversity gain of the repetition code affect its codeword error probability. (NB: We do not consider the fact that the codes have different rates in the comparison.)

3. In section 2.3 the normalization for the noise variance was given. Using the channel assumptions given in this section:
2.7. PROBLEMS

a) Derive the energy normalization equations as a function of $E_b/N_0$ for the following channel scenarios:
   i) the SISO channel model according to equation 2.5,
   ii) the MISO channel model according to equation 2.6,
   iii) the SIMO channel model according to equation 2.7.

b) Based on the derivations, offer an explanation as to why the noise variance is the same for the SISO and the SIMO channels but different for the MISO channel.

Hint: Start by writing the SNR of the received signal $r$ according to equation 2.3 and solve for the noise variance $\sigma_n^2$ for the particular scenario. Use the technique given in [Pro95]. Here the expectation value of the signal $s$ is found, assuming that the channel coefficient $h$ is constant. Next, the expectation of the result is taken with respect to the channel coefficient.

4. Based on the theory section, what are the major advantages and disadvantages of the spatial MIMO system in comparison with temporal SISO transmission?
Chapter 3

MIMO codes and signal processing techniques

3.1 Beamforming

Beamforming is a signal processing technique which exploits the spatial geometry of the received signal to add the energy collected from the received antennas coherently [KV96]. In this way, the concept of beamforming is the same as for MRC equalization. Figure 3.1 shows the spatial geometry of the receiver antenna array in the far field. We consider only the linear antenna array geometry, and assume that the channel coefficient $h$ and direction of arrival (DOA) $\theta$ is perfectly known at the receiver. Separation among antenna elements is based on the wavelength of the transmitted signal $\lambda$. In the following we consider receiver beamforming. However, based on the principle of reciprocity, the same considerations may be made for transmit beamforming.

![Figure 3.1: Beamforming system](image-url)
3.1.1 Beamforming channel

As shown in figure 3.1, we consider the reception of the transmitted signal in the far field. This validates the assumption that the received electromagnetic energy is a plane wave impinging on the receiver array. Furthermore, the receive antennas are positioned in close proximity and we assume a quasi-static, narrow band channel model in the time and frequency dimensions. Ideally, the distance between the antennas is $\lambda/2$, where $\lambda$ is the wavelength of the transmitted signal. This distance can be thought of as the Nyquist sampling rate in the spatial dimension. Because the antennas are close together, they are spatially correlated. When we consider the channel diversity based on section 2.1, we have the result

$$
\mathbb{E}_h \{ \mathbf{H} \cdot \mathbf{H}^H \} = \left[ \begin{array}{c}
\mathbf{U} \\
\mathbf{0}
\end{array} \right] \cdot \left[ \begin{array}{c}
\lambda_1 \\
\vdots \\
0
\end{array} \right] \cdot \left[ \begin{array}{c}
\mathbf{U} \\
\mathbf{0}
\end{array} \right]^H.
$$

(3.1)

Therefore the spatial channel diversity is one in this system, and all antennas are correlated. Due to the quasi-static narrow band nature of the channel, there is no diversity in the time or frequency dimensions.

3.1.2 Calculation of antenna weights, $W$

Because all antennas are correlated, the spatial geometry of the array can be used to perform MRC combining of the collected energy. Only one channel coefficient must be estimated, because the remaining $N_{rx} - 1$ coefficients may be determined based on the antenna array geometry. For the antenna array in figure 3.1 we normalize the wavelength to $\lambda = 1$. The antenna weights are

$$
W_k = h^* \cdot \exp \left\{ -j2\pi \cdot \left( k - 1 \right) \cdot d \right\} \quad (3.2)
$$

$$
= h^* \cdot \exp \left\{ -j2\pi \cdot \left( k - 1 \right) \cdot \frac{d}{\lambda} \cdot \cos(\theta) \right\}.
$$

Each $W_k$ receives the transmitted signal at a different time instant, because some signals must travel a distance of $k \cdot d$ further, as shown in figure 3.1. The final received signal for a single time instant is given as

$$
r = N_{tx} \cdot |h|^2 \cdot s + \tilde{\eta},
$$

(3.3)

where $\tilde{\eta} = \sum_{j=1}^{N_{tx}} W_j \cdot \eta_j$ is the sum of AWGN noise terms. It is also AWGN with mean $\mu_{\tilde{\eta}} = 0$ and variance $2\sigma_{\tilde{\eta}}^2 = N_{tx} \cdot 2\sigma_n^2$.

The principle of beamforming is relatively simple, however the implementation of such a system has several practical difficulties. The antenna must be calibrated for a particular wavelength. Only one channel coefficient must be calculated but the DOA must also be determined and, for time varying channels, tracked. This requires either pilot sequences, or a feedback channel. Furthermore, equation 3.1 shows that the channel has no diversity as defined in section 2.1.

3.1.3 Beam pattern

Two types of beam patterns may be plotted. The first is the amplitude of the antenna gain with respect to the angle. Usually the logarithm of the antenna gain is taken. The second plot is the beam pattern. For this result, the antenna amplitude is again plotted with respect to the angle, but this time in terms of polar coordinates. This may be done using the Matlab function $\text{polar}$. Figure 3.2 shows examples of these plots.
3.2 ORTHOGONAL SPACE TIME CODES

Orthogonal space time codes map modulated symbols to a matrix which is transmitted over the $N_{tx}$ statistically independent antennas using $N_t$ transmission periods [Ala98] [TJC99]. The codeword matrix is constructed such that every row is orthogonal in the complex field, independent of the symbols which are transmitted. Figure 3.3 shows the orthogonal space time code system. The orthogonal space time code design exploits the maximum diversity of the system for high $E_b/N_0$ channels. Due to the orthogonal property of the codewords, the codes are guaranteed to have diversity equal to $N_{tx}$.

The major advantage of these codes is that they are simple to decode, and are applicable for any signal constellation. Their disadvantages include the fact that there is a temporal rate loss $R_t < 1$ when $N_{tx} > 2$ and the codes exist only for a limited number of matrix dimensions. The coding gain for these designs is restricted by the orthogonality requirement and the signal constellation. Furthermore, the performance is sensitive to the accuracy of the channel coefficient estimation.

Figure 3.3: Orthogonal space time code system
3.2.1 Orthogonal space time code channel

The channel assumptions for orthogonal space time codes are given in section 2.2. The channel correlation matrix for a single receive antenna is

$$\mathbb{E}_h \left\{ \mathbf{h}^H \cdot \mathbf{h} \right\}^{\text{Orth.}} = \left[ \begin{array}{c} \mathbf{U} \\ \mathbf{0} \end{array} \right] \left[ \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_{N_{tx}} \end{array} \right] \left[ \begin{array}{c} \mathbf{U} \\ \mathbf{0} \end{array} \right]^H,$$

where all eigen values are non-zero. Therefore the channel diversity is equal to $N_{tx}$ in contrast to the beamforming channel which has a spatial diversity of one. Due to the quasi-static, narrow band nature of the channel, there is no diversity in the time or frequency dimensions. The orthogonal space time code allows the exploitation of the full diversity of the channel.

3.2.2 Orthogonal space time code templates

There are several orthogonal matrix designs which are commonly used in the literature. The following matrices are used as templates for the construction of the codeword matrix. i.e the variables $x_i \in \mathcal{A}$. $N_{tx}$ and $N_t$ are the number of transmit antennas and the number of timeslots of the code respectively. The template matrices are given in the following table:

<table>
<thead>
<tr>
<th>Code</th>
<th>$N_{tx} \times N_t$</th>
<th>rate $R_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2^\perp$</td>
<td>2 $\times$ 2</td>
<td>1</td>
</tr>
<tr>
<td>$S_3^\perp$</td>
<td>3 $\times$ 8</td>
<td>1/2</td>
</tr>
<tr>
<td>$S_{4a}^\perp$</td>
<td>4 $\times$ 4</td>
<td>3/4</td>
</tr>
<tr>
<td>$S_{lb}^\perp$</td>
<td>4 $\times$ 4</td>
<td>1/2</td>
</tr>
</tbody>
</table>

Table 3.1: Orthogonal code parameters

$$S_2^\perp(\vec{x}) = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}, \quad (3.4)$$

$$S_3^\perp(\vec{x}) = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & x_1^* & -x_2^* & -x_3^* & -x_4^* \\ x_2 & x_1 & -x_4 & x_3 & x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & -x_4 & x_1 & x_2 & x_3^* & -x_4^* & x_1^* & x_2^* \end{bmatrix}, \quad (3.5)$$

$$S_{4a}^\perp(\vec{x}) = \begin{bmatrix} x_1 & -x_2^* & x_3 & 0 \\ x_2 & x_1^* & 0 & x_3 \\ x_3 & 0 & -x_1^* & -x_2^* \\ 0 & x_2 & x_3 & -x_1 \end{bmatrix}, \quad (3.6)$$

$$S_{lb}^\perp(\vec{x}) = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & x_1^* & -x_2^* & -x_3^* & -x_4^* \\ x_2 & x_1 & -x_4 & x_3 & x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & -x_4 & x_1 & x_2 & x_3^* & -x_4^* & x_1^* & x_2^* \\ x_4 & x_3 & -x_2 & x_1 & x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix}. \quad (3.7)$$

We emphasize that for the normalized noise variance using orthogonal space time codes (section 2.3), the rate $R_t$ is not necessarily equal to one.
### 3.2.3 Orthogonal MRC decoding

The decoding of the orthogonal codes allows for the recovery of the individual signals as a function of each channel. We consider as an example the recovery of the vector \( \mathbf{x} = (x_1, x_2) \) of the code \( S_1^4 \):

\[
\hat{x}_1 = h_1^* \cdot r_1 + h_2 \cdot r_2^* \\
= (|h_1|^2 x_1 + h_1^* h_2 x_2 + h_1^* \eta(1)) + (|h_2|^2 x_1 - h_1^* h_2 x_2 + h_2 \eta(2)) \\
= (|h_1|^2 + |h_2|^2) x_1 + h_1^* \eta(1) + h_2 \eta(2) \\
= (|h_1|^2 + |h_2|^2) x_1 + \tilde{\eta}_1 \\
\hat{x}_2 = h_2^* \cdot r_1 - h_1 \cdot r_2^* \\
= (|h_2|^2 x_2 + h_2^* h_1 x_1 + h_2^* \eta(1)) + (|h_1|^2 x_2 - h_2^* h_1 x_1 + h_1 \eta(2)) \\
= (|h_1|^2 + |h_2|^2) x_2 + h_1^* \eta(1) - h_1 \eta(2) \\
= (|h_1|^2 + |h_2|^2) x_2 + \tilde{\eta}_2.
\]

Here we see that the estimates \( \hat{x}_1 \) and \( \hat{x}_2 \) are dependent on the coherent addition of the two channel coefficients. There are also two noise terms \( \eta(1), \eta(2) \) for each recovered signal which result from the combination of two received signals. The noise terms \( \tilde{\eta}_i \) are AWGN with variance \( 4\sigma_n^2 \). The recovery of the transmitted signals for other orthogonal codes may be done similarly by considering a combination of the sum of the received signals \( r_i \), \( 1 \leq i \leq N_t \).

### 3.3 Problems

1. a) Derive the \( SNR_{SIMO} \) as given in figure 1.3, i.e. the sum of signals for each antenna directly after the noise term. This result was part of the solution for problem 3.c) for the SIMO channel.

b) Derive the \( SNR_{MRC} \) as given in figure 2.2, i.e. the sum of signals for each antenna after multiplication by the channel coefficients.

c) Derive the \( SNR_{BF} \) as given in figure 3.1, i.e. the sum of signals for each antenna after multiplication by the beam weights.

d) From b) and c) we find that \( SNR_{BF} = SNR_{MRC} \). Based on the channel assumptions specific to each system, do you expect them to have the same error rate performance? Hint: Consider the channel diversity of both systems, and the solution to problem 2.2.

2. Show that the code \( S_4^4 \) has full rank and therefore can achieve a the full channel diversity of 4. Hint: Find the auto correlation of the code template.
Chapter 4

Simulation tips

In the exercises which follow, it will be necessary to simulate several different transmission scenarios. This chapter offers some suggestions on the program structure for the different exercises. Any simulation tool may be used i.e. Matlab, Co-centric, C, etc, however, there will be some specific examples given using the Matlab programming environment.

The generic transmission system is given in figure 4.1 for the MIMO channel model. We next describe these functional blocks in detail.

![Transmission System Diagram](image)

**Figure 4.1: Transmission system**

### Simulation chain

**Parameters:**
These are the variables which will be changed for the different exercises, i.e. it is suggested that they be given variable names to simplify the modification of the program.

- $N_{tx}$, $N_{rx}$: Number of transmit and receive antennas
- $N_{cw}$: This variable determines the number of codewords which will be transmitted in the simulation. For good results, usually 100 codewords should be received incorrectly. Therefore, for a particular codeword error probability $P_{cw}$, the number of codeword needed for good simulation statistics can be found.

$$N_{cw} = \frac{100}{P_{cw}}$$

- $N_i$: The number of information symbols. This value is usually determined by the number of codewords.

$$N_i = N_{cw} \cdot R_i$$
$E_b/N_0$: The information bit energy to noise ratio. This parameter in dB determines which points along the x-axis of the simulation plot are used. It is usual to increment the values by 3 dB, which is equivalent to doubling the signal energy.

$\sigma_n^2$: The noise variance gives the energy of the AWGN noise sources at each receive antenna. The noise variance is given as a function of $E_b/N_0$ for different scenarios in section 2.3. Note: the $E_b/N_0$ must be converted from dB to a power ratio for the simulation environment, i.e the SNR must be found as a function of $\sigma_n^2$.

**Source:**

**Function:** The source generates the information symbols. These values are usually given as integer numbers from 0, 1, \ldots, $|A| - 1$, where $|A|$ is the cardinality of the signal constellation.

**Output:** A vector of integers from 0, 1, \ldots, $|A| - 1$. An example Matlab command to generate $N_i$ equally probable random symbols is:

$$\vec{r} = \text{floor}(\text{rand}(N_i, 1) * |A|)$$

**Modulation:**

**Input:** The information symbol vector $\vec{r}$ from the source.

**Function:** Mapping of from the integers 0, 1, \ldots, $|A| - 1$ to a complex signal constellation $\mathcal{A}$. Two examples are given for BPSK and QPSK modulation in Matlab. Here we assume the information vector is a series of ones and zeros for both BPSK and QPSK.

$$\vec{r} \rightarrow_{\text{BPSK}} \vec{x} = (-1)^{\vec{r}} \quad \vec{r} \rightarrow_{\text{QPSK}} \vec{x} = ((-1)^{\vec{r}_{\text{odd}}} + j(-1)^{\vec{r}_{\text{even}}}) \cdot \exp\left(\frac{\pi}{4}\right)$$

**Output:** A vector of complex information symbols $\vec{x} \in \mathcal{A}$.

**Coding:**

**Input:** Vector of information symbols $\vec{r}$ or modulation symbols $\vec{x}$.

**Function:** Mapping from the information / modulation symbols to a complex codeword $s \in \mathcal{S}$ or an integer codeword $c \in \mathcal{C}$ with a particular rate $R_i$, distance $d$ and cardinality $|\mathcal{S}|$ or $|\mathcal{C}|$.

**Output:** A codeword with symbols from a complex signal constellation or set of integers. For multiple input systems this is usually a matrix of size $N_{tx} \times N_i$. An example of encoding a BPSK vector $\vec{x}$ to the orthogonal code $\mathcal{S}_2^+$ from section 3.2 is as follows:

$$\vec{r} = (\ell_1, \ell_2) = (0, 1) \rightarrow_{\text{BPSK}} \vec{x} = ((-1)^{\ell_1}, (-1)^{\ell_2}) = ((-1)^0, (-1)^1) = (1, -1) \rightarrow s_{2}^{+}$$

$$\mathbf{s} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

**Channel:**

**Input:** A transmitted sequence consisting of the complex codewords $s$ or a complex vector of modulation symbols $\vec{x}$.

**Function:** Multiplies the elements of the transmitted sequence by the channel coefficient which has a particular distribution, and statistical properties. For the constant channel, all coefficients are 1. The assumptions for the channel coefficients for the MIMO system considered in this lab are given in section 2.2.
Output: Either a vector or matrix of random complex signals which depend on the particular statistical channel model, i.e. the statistics for the Rayleigh channel are given in equation 2.2.

Receiver:

Input: The transmitted sequence weighted by the channel coefficients plus the noise term.
Function: Models the physical receive antenna which gathers the transmitted energy. Each received signals is corrupted by a noise source. The mathematical description of the received signal depends on the channel model, and the channel assumptions.

Output: A noise corrupted version of the transmission sequence is output. The variance of the noise $\sigma^2$ depends on the channel model as described in section 2.3. Note that to modify the distribution of a set of AWGN random variables, we must multiply them by the square root of the variance [Pap84]. For example

$$\eta : N(0, 1) \rightarrow \hat{\eta} = \sqrt{\sigma^2} \eta : N(0, \sigma^2),$$

where we have changed the Gaussian random variable $\eta$ with a variance of one to a Gaussian random variable $\hat{\eta}$ with a variance of $\sigma^2$. An example of the MISO channel receiver for the orthogonal code $S_{12}$ under the channel assumptions of section 2.2 is

$$\bar{r} = h * s + \eta = [h_1, h_2] \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} + [\eta(1), \eta(2)].$$

Equalization:

Input: Output signal from the receiver $\bar{r}$.
Function: Based on the channel and receiver properties, the equalizer coherently adds the signal energy, and removes interference.

Output: A signal $\tilde{r}$ with an improved signal to noise ratio. MRC and beamforming from sections 2.4 and 3.1 respectively are two examples of equalization.

Demodulation:

Input: The signal from the receive antenna $\bar{r}$.
Function: Estimates the transmitted signal based on the received signal. Different methods may be used such as maximum likelihood demodulation as described in section 2.5.

Output: The signal point which has the closest distance to the received signal based on an estimate of the channel coefficients is assumed to be the most probable. Useful Matlab commands for calculating the maximum likelihood value of the received signal include $\text{ kron(\cdot)}$, $\text{ min(\cdot)}$, $\text{ abs(\cdot)}$ and $\text{ sum(\cdot)}$. An example of maximum likelihood demodulation for a BPSK MRC equalized signal $\hat{r}$ is given as follows

$$\hat{x} = \min \left\{ \left\| \begin{bmatrix} |h|^2 x_1 \\ |h|^2 x_{-1} \end{bmatrix} - \begin{bmatrix} \hat{r}^2 \\ \hat{r} \end{bmatrix} \right\| \right\} 
= \min \left\{ \left\| \begin{bmatrix} |h|^2 x_1 + h^* \eta \\ |h|^2 x_{-1} + h^* \eta \end{bmatrix} - \begin{bmatrix} |h|^2 x_1 \\ |h|^2 x_{-1} \end{bmatrix} \right\| \right\} 
= \min \left\{ \left\| |h|^2 (x_1 - x_{-1}) + h^* \eta \right\| \right\}$$

Decoding:
**Input:** Depending on the system, either the receiver output \( \hat{r} \), equalizer output \( \hat{f} \) or the demodulation output \( \hat{x} \).

**Function:** Maps the input signal to a codeword \( s \) or \( c \) for complex, and real codes respectively. The maximum likelihood technique from section 2.5 may be used for optimum decoding.

**Output:** Either the estimated codeword \( \hat{s}, \hat{c} \) or the estimated information sequence \( \hat{r} \). The same principle for demodulation is used where codewords replace the modulation signals.

**Error statistics:**

**Input:** The estimate of the modulation sequence \( \hat{x} \), the codeword sequence \( \hat{s} \), or the information sequence \( \hat{r} \).

**Function:** Compares the estimate at the receiver to the transmitted signal. Calculates the error probability. This is often in terms of bit, symbol, or codeword error.

**Output:** The error statistics which are plotted against the associated signal to noise ratio. Useful Matlab functions include \( \text{mean}(\cdot) \), \( \text{sum}(\cdot) \), and the boolean operations. An example of the error statistic calculation for a sequence of information in Matlab is given below:

\[
\text{err}_{cw} = \text{mean}(\hat{r} \sim \hat{r})
\]

**Program structure**

A suggested structure for the program is given in the following. In this format, all \( N_{cw} \) are simulated for one \( E_b/N_0 \) value, and then saved. Therefore the program generates a separate file for each \( E_b/N_0 \) value. The advantage with this format is that individual simulation points may be easily evaluated. The disadvantage is, is that the user has more difficulty plotting and organizing the files. Note: a modulation symbol can be considered as a codeword of length one.

**Parameters**

**Initialization**

**E\(_b\)/N\(_0\) Loop**

**Codeword loop**

- Generate information vector
- Modulation / Encoding
- Generate channel coefficients and noise terms
- Receiver simulates the transmission over the channel
- Demodulation / Decoding
- Calculation of error statistics for the codeword.

**End codeword loop**

- Save statistics for \( E_b/N_0 \) loop

**End E\(_b\)/N\(_0\) Loop**

**Beam pattern antenna plots**

To plot the beam pattern of an antenna array, first the antenna weights \( W \) must be calculated for the given DOA \( \theta \). Then the antenna gain is calculated for the range of angles 0 to 2\( \pi \). Figure 6.5 shows the principle behind the beam pattern plots. The \( X \) coordinates on the circle indicate discrete angles of arrival from 0 to 2\( \pi \). The resulting antenna gain will be largest when the angle \( X \) is equal to the DOA.
Figure 4.2: Plotting of the beam pattern
Chapter 5

Exercises:

In the following exercises, you will be required to simulation the bit error performance for different MIMO systems. The same program may be used for all exercises, under the condition that the parameters such as the number of transmit, receive antennas is variable, and that the channel statistics may be modified for constant and Rayleigh statistics.

5.1 Single transmit/receive antenna system

This exercise offers a simple transmission model that will be used as a basis for the following exercises. The system has a SISO channel (section 1.1) and is shown in figure 5.1. The information symbols are modulated, and transmitted over the channel. AWGN noise $\eta$ is added at the receiver. The received symbol is then equalized based on MRC (section 2.4) and demodulated based on the ML principle over all possible transmit symbols (section 2.5).

Table 5.1 includes the associated parameters and simulation ranges for the system.

<table>
<thead>
<tr>
<th>Channel</th>
<th>Code Type</th>
<th>$N_{tx}$</th>
<th>$N_{rx}$</th>
<th>$\mathcal{A}$</th>
<th>Min. BER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant = 1</td>
<td>None</td>
<td>1</td>
<td>1</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>None</td>
<td>1</td>
<td>1</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters for the single transmit / receiver system.

![Figure 5.1: Single transmit / receive antenna system](image)

a) Find the bit error rate of the SISO system for an AWGN channel with constant channel coefficients. Use the AWGN error curve from section 2.5 to verify that your program and normalization is correct before simulating the Rayleigh channel. The programs which generate and plot this curve are called $\text{ex}_\text{awgn}_\text{channel.m}$ and $\text{ex}_\text{awgn}_\text{channel}_\text{plot.m}$ respectively, and are found in the directory $\text{ex}_\text{awgn}_\text{channel}$. These programs may be used as models for your investigations.
b) Calculate the slope of the constant and Rayleigh simulation results based on equation 2.1.

c) Explain qualitatively why the performance of the two curves are different. Hint: Consider your answer from problem 1 of section 2.7.
5.2 Maximum ratio combining (MRC) receiver

In the following, the performance of a MRC equalizer (section 2.4) with multiple antennas will be investigated and compared with the single transmit / receive antenna system from example 5.1. The system model is given in figure 5.2.

Table 5.2 includes the associated parameters and simulation ranges for the system.

<table>
<thead>
<tr>
<th>Channel</th>
<th>Code Type</th>
<th>$N_{tx}$</th>
<th>$N_{rx}$</th>
<th>$\mathcal{A}$</th>
<th>Min. BER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant = 1</td>
<td>None</td>
<td>1</td>
<td>${2, 4}$</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>None</td>
<td>1</td>
<td>${2, 4}$</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.2: Parameters for the MRC receiver system.

![Figure 5.2: Maximum ratio combining (MRC) receiver](image)

a) Modify your program from section 5.1 to simulate the MRC multiple receive antenna system. Use the noise normalization as described for the SIMO channel in section 2.3.3 and the receiver structure as described in section 2.4.

b) Calculate the slope of the constant and Rayleigh channel curves. Explain the similarities and differences of the SIMO system and the SISO system of section 5.1 with respect to their slope and antenna gain. Use the theory from chapter 2.

c) If we let $N_{rx} \to \infty$, what would be the antenna gain, and the bit error probability? Is this realistic, why or why not?
5.3 Multiple transmit / single receive antenna system

In the following, the performance of a multiple transmit antenna system will be investigated and compared with the single transmit / receive antenna system from exercise 5.1. The same modulated information is transmitted over all antennas, and the superposition of the signals are obtained at the receiver. Based on the superposition property, a type of MRC equalization is done. The system model is given in figure 5.3.

Table 5.3 includes the associated parameters and simulation ranges for the system.

<table>
<thead>
<tr>
<th>Channel</th>
<th>Code Type</th>
<th>$N_{tx}$</th>
<th>$N_{rx}$</th>
<th>$\mathcal{A}$</th>
<th>Min. BER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant = 1</td>
<td>None</td>
<td>{2, 4}</td>
<td>1</td>
<td>BPSK</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>None</td>
<td>{2, 4}</td>
<td>1</td>
<td>BPSK</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.3: Parameters for the MRC receiver system.

![Figure 5.3: MISO system](image)

Figure 5.3: MISO system

a) Using your program from the previous sections, modify it to simulate the MISO system. The same signal is transmitted over all antennas. The noise normalization should be done according to section 2.3.2. In the receiver, use the MRC-like equalization for the MISO system as described in section 2.4.

b) Compare the slope of the simulation curves with those from exercise 5.1, and comment on the similarities and differences using the theory from chapter 2.

c) What is the antenna gain of this system. What happens to the bit error rate if $N_{tx} \to \infty$? How is this result different than that for the SIMO system when $N_{rx} \to \infty$? Hint: Consider the derivation of the noise normalization for the SIMO and MISO systems.
5.4 Orthogonal space time codes

The next investigation will focus on the encoding and simulation of orthogonal space time codes as presented in section 3.2. The system modes is given in figure 5.4. The receiver uses the orthogonal properties of the code to recover the individual information symbols which are then demodulated to give the information symbols.

Table 5.4 includes the associated parameters and simulation ranges for the system.

<table>
<thead>
<tr>
<th>Channel</th>
<th>Code Type</th>
<th>$N_{tx}$</th>
<th>$N_{rx}$</th>
<th>$N_{t}$</th>
<th>$\mathcal{A}$</th>
<th>Min. BER</th>
<th>Min. CER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant = 1</td>
<td>Orth. 2x2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>BPSK</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>Orth. 2x2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>BPSK</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.4: Parameters for the orthogonal space time code.

Figure 5.4: Orthogonal space time code system

a) Write a simulation program for the space time orthogonal code based on the description in section 3.2. Use the noise energy normalization according to section 2.3.2. What is the temporal rate of the code $R_t$, i.e. how many information symbols are transmitted per timeslot? Find both the bit and the codeword error probability of the system.

b) Calculate the slope of both the constant and Rayleigh channels. Compare this result to the slopes from section 5.1 and explain the results based on the channel model, and the code design, i.e. contrast the orthogonal code performance to the MISO results.

c) Does the orthogonal code have an antenna gain for the different channels? Support your answer based on the noise normalization. Hint: Compare the results to the MRC simulations.
5.5 Beamforming system

The principles of beamforming were presented in section 3.1. Symbols are transmitted from a single antenna and are received by multiple correlated antennas in a linear array which have a separation of $1/2\lambda$ wavelengths. The receiver performs the appropriate equalization such that the symbols from each antenna branch can be added coherently and demodulated based on the direction of arrival (DOA) angle $\theta$.

Table 5.5 includes the associated parameters and simulation ranges for the system.

<table>
<thead>
<tr>
<th>Channel</th>
<th>Sep.</th>
<th>DOA $\theta$</th>
<th>Code Type</th>
<th>$N_{tx}$</th>
<th>$N_{rx}$</th>
<th>$A$</th>
<th>Min. BER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant = 1</td>
<td>$1/2\lambda$</td>
<td>$\pi/4$ rad</td>
<td>None</td>
<td>1</td>
<td>${2, 4}$</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>$1/2\lambda$</td>
<td>$\pi/4$ rad</td>
<td>None</td>
<td>1</td>
<td>${2, 4}$</td>
<td>$BPSK$</td>
<td>$10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.5: Parameters for the Beamforming system.

Figure 5.5: Beamforming system

a) Based on the simulation tips from section 4, find the beam pattern for the given angle $\theta$ from 0 to $2\pi$. What happens to the beam pattern as the number of antennas increases. Comment on the advantages and disadvantages associated with the beamforming system over an omni directional antenna.

b) Calculate the antenna coefficients $W_i$ based on the given DOA $\theta$ and write a program which simulates the bit error rate of the system for the constant and the Rayleigh channel. Choose the correct energy normalization based on section 2.3.

c) What is the channel correlation matrix of this system, and what is its associated diversity for the constant and Rayleigh channels. Is it possible for this beam forming system to have a diversity gain?

d) What is the antenna gain of the different systems? What are the major differences between a beamforming and a MRC receiver?
Chapter 6

Solutions to the exercises

Exercise 5.1: Single transmit / receive antenna system

a) Figure 6.1 shows the error performance curves for the constant and Rayleigh channels with AWGN noise.

![BPSK AWGN/Rayleigh Channel Performance Graph](image)

Figure 6.1: SISO error performance for constant and Rayleigh amplitude channels.

b) Based on equation 2.1, we get the following slopes:

\[ \Gamma_{\text{constant}} \approx 4.3/dB \quad \Gamma_{\text{Rayleigh}} = 1.0/dB \]

The slope is independent of the signal constellation used. Note that for the range of values taken, the constant channel has not reached its asymptotic value.

c) In section 2.7 the error performance of an erasure channel was considered. The Rayleigh channel may be considered in a similar way. Sometimes the channel coefficient \( h \approx 0 \) which will
result in a large probability of error, independent of the transmit $E_b/N_0$. This is similar to an erasure. In contrast, when transmitting a signal over a channel with a constant amplitude, and AWGN noise is only dependent on the $E_b/N_0$ ratio, i.e. there is no erasure effect due to the channel.

**Exercise 5.2: Maximum ratio combining (MRC) receiver**

a) The simulation results for the constant and Rayleigh channel are given in figure 6.2.

![Simulation Results](image)

(a) Constant Channel  
(b) Rayleigh Channel

Figure 6.2: SIMO error performance for constant (a) and Rayleigh (b) amplitude channels.

b) The slope of the constant curves from the SIMO and the SISO system are the same. This is to be expected because for the SIMO system all channels are correlated, and therefore the channel correlation matrix for the SISO and the SIMO system have one eigen value. The SIMO system has an antenna gain of $N_{rx}$ due to the coherent addition of the channel paths.

The slope of the Rayleigh bit error performance for the SIMO and the SISO systems are different. The SIMO system has 2 and 4 uncorrelated paths which corresponds to a channel diversity of 2 and 4 according to section 2.1. In addition to the diversity gain of the SIMO system, the MRC introduces an antenna gain of $N_{rx}$ because of the coherent addition of the signal energy.

c) If $N_{rx} \to \infty$, then the antenna gain would be infinite, and the bit error probability would be zero, independent of the signal to noise ratio of the system. This is not a realistic result. The energy input into the system is finite, i.e. $|s| = 1$ per timeslot. The difficulty arises in the simple mathematical model which we use, which does not account for the electromagnetic propagation of the signal.

**Exercise 5.3: Multiple transmit / single receive antenna system**

a) The simulation results for the constant and Rayleigh channel are given in figure 6.3.
b) The slope for the constant channel is the same for the SISO and the MISO system. This can be seen by calculating the channel correlation matrix. For both cases there is only one non-zero eigen vector. The MISO system shows an antenna gain of $N_{tx}$. This is due to the fact that the channels are added coherently, despite the super position at the receiver, i.e. both channel amplitudes are always 1, and therefore have a coherence value of 1. The noise normalization equations were derived under the assumption that the channel coefficients were independent random processes.

The SISO and MISO channel is also the same for the Rayleigh channel. In this case the MISO channel correlation has $N_{tx}$ non-zero eigen values, however, the receiver is not able to exploit this channel diversity. The transmitted signal is super imposed at the receiver, and cannot be separated for MRC decoding as in the SIMO case. In order to exploit the diversity, some sort of code, or signal property must be used to allow the receiver to separate the transmission paths.

c) There is no antenna gain in the system. The number of transmit antennas does not give an antenna gain. The reason for this can be seen from the derivation of the noise normalization from section 2.3.2. Here, the increase in the number of transmit antennas is canceled in the noise normalization due to the assumption that $|s| = 1$ per timeslot. In contrast, the noise normalization of the SIMO system is independent of $N_{rx}$.

**Exercise 5.4: Orthogonal space time codes**

a) The simulation results for the constant and Rayleigh channel are given in figure 6.4.

b) The constant channel has the same slope as for the SISO system. For the channel correlation matrix, there is only one non-zero eigen value, which is also the case for the SISO channel. The slope of the Rayleigh channel shows a diversity gain which is the same as that for the SIMO channel with the same number of antennas. Therefore, the orthogonal code exploits the diversity of the channel. This is possible due to the orthogonal property of the codewords. The MISO system could not exploit the channel diversity because the super position of the signals
could not allow the equalization of the different channels separately i.e. see the equations from section 3.2.3.

c) There is an antenna gain for the constant channel for the same reasons as given in exercise 5.3, i.e. the signals from the transmit antennas are added coherently.

There is no antenna gain for the Rayleigh channel. This is due to the energy normalization at the transmitter which ensures a transmit energy of one per timeslot. The arguments are the same as for the MISO channel.

**Exercise 5.5: Beamforming system**

a) Figure 6.5 shows the beam pattern for different numbers of antennas. As the number of antennas increases, the beam becomes narrower. This has the advantage that the antenna gain increases, and that the interference to other users in the system decreases. In contrast, the omni directional antenna radiates its signal in all directions, which causes interference to other system users. The difficulty in having a narrow beam pattern arises when the target user is moving. A narrow beam would be more sensitive to tracking errors.

b) The channel coefficients may be calculated based on equation 3.3 from section 3.1. The noise normalization is the same as for the SIMO scenario. Figure 6.6 shows the bit error rate for the different number of beamforming antennas.

c) The channel correlation gives a single eigen value, and therefore has a diversity of 1 for both the constant and Rayleigh channels. With this beamforming system model, it is not possible for the receiver to have a diversity gain. This is because the receiver requires that the signals at each antenna be correlated. Then, based on the antenna geometry the energy can be coherently added according to equation 3.3. There are beamforming systems which can achieve a limited type of diversity gain by dividing the receiver antenna array into sub arrays, where the channel coefficients among the sub arrays are uncorrelated.
d) The beamforming system has an antenna gain equal to the number of receive antennas. The major difference in comparison to the MRC receiver, is that the beamforming system requires that the signals at the receiver are correlated. Only one channel coefficient must be estimated as compared to $N_{rx}$ channel coefficients for the MRC system.
Bibliography


