

Schur-Based Decomposition for Reachability Analysis of Linear Time-Invariant Systems

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Abstract—We present a method for complexity reduction in reachability analysis and controller synthesis via a Schur-based decomposition for LTI systems. The decomposition yields either decoupled or weakly-coupled subsystems, each of lower dimension than the original system. Reachable sets, computed for each subsystem, are back-projected and intersected to yield an overapproximation of the actual reachable set. Evaluating our method for a variety of examples (3D, 4D, and 8D), we show that significant reduction in the computational costs can be achieved. This technique has considerable potential utility for use in conjunction with computationally intensive reachability tools.

Keywords: reachability analysis, dimension reduction, projection, LTI systems, decomposition

I. INTRODUCTION

Reachability analysis is key for verification and controller synthesis of continuous and hybrid dynamical systems, yet a major obstacle in employing reachability analysis is the “curse of dimensionality” [1]. The computational complexity of reachability techniques scales poorly with the dimension of the continuous state space, often rendering them impractical for complex real-life applications. While efficient reachability techniques have been developed recently [2], [3], their utility is restricted to systems whose constraints can be described by specific classes of shapes (e.g., ellipsoids and zonotopes) in both the input and the state spaces. For some applications, however, it is crucial to be able to take advantage of some of the unique features (e.g., safety controller synthesis, and handling of non-convex or arbitrarily shaped sets) offered almost exclusively by more computationally intensive reachability tools.

This paper focuses on continuous linear time-invariant (LTI) systems (and by extension, hybrid systems with LTI continuous dynamics). We aim to broaden the range of applicable reachability tools for LTI systems with high dimensionality, to enable the use of reachability tools that would otherwise be too computationally complex to employ (e.g., [4], [5] and [6]). We accomplish this through the use of a Schur-based decomposition, inspired by a model reduction algorithm for systems with unstable modes [7], [8].

Our method decomposes LTI systems into either completely decoupled or weakly-coupled subsystems. Instead

of solving one high-dimensional problem, multiple lower-dimensional problems are solved, hence reducing complexity regardless of the reachability tool used. Reachability analysis can be performed on each subsystem independently. Back projecting and intersecting each of the lower-dimensional reachable sets provides an overapproximation of the actual reachable set. A Sylvester equation (or an optimization problem) is solved in order to eliminate (or minimize) the coupling between the subsystems. Additional constraints are imposed when the control input is non-disjoint across subsystems, to prevent underapproximation of the reachable set. By performing reachability on these lower dimensional subsystems we obtain significant reduction in the computational costs, albeit at the expense of overapproximation.

Complexity reduction for reachability analysis is well-studied and techniques to compute reachable sets for higher dimensional systems can be divided into three categories: (i) techniques that take advantage of specific representations of sets in the state space [2], [3], [9], (ii) techniques that make use of model reduction and approximation [10], [11], hybridization [12], projection [13] and structure decomposition [14], [15], and (iii) techniques that combine the approaches from the first two categories. For instance, [16] employs both model approximation (through Krylov subspace projection) and efficient set representation (using low-dimensional polytopes) to perform reachability for very large-scale systems with affine dynamics.

In [14], a full-order nonlinear system is decomposed to either disjoint or overlapping subsystems and multiple Hamilton-Jacobi-Isaacs PDEs are solved in lower dimensions. The computed reachable set for each subsystem is an over-approximation of the projection of the full-order reachable set onto the subsystem’s subspace. In [15], using an ϵ -decomposition procedure, affine systems are decomposed into multiple subsystems and reachability is performed on each lower-dimensional subsystem. Specifically, given a system $\dot{x} = Ax + b$, the matrix A is written as $A = A_D + \epsilon A_C$, where A_D is block diagonal. These blocks are decoupled, as the coupling between them is captured in ϵA_C . Reachability is then performed for each isolated subsystem $\dot{x}_{Di} = A_{Di} x_{Di} + b_i$; if the computed error (introduced by discarding the coupling between subsystems) is too large, higher order subsystems are considered and the procedure is repeated.

Our main contribution is to provide an additional method to reduce the complexity of reachability analysis for high dimensional LTI systems through decomposition of system dynamics. In Section II, we formulate the decomposition

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problem for LTI continuous systems and provide necessary mathematical preliminaries. Section III presents the decomposition method for two cases: decomposition that results in a) decoupled subsystems, or in b) weakly-coupled subsystems. An extension to hybrid systems is also provided. Section IV demonstrates our method on several numerical examples, in 3D, 4D, and 8D. Lastly, we provide conclusions and directions for future work in Section V.

II. PROBLEM FORMULATION AND MATHEMATICAL PRELIMINARIES

Consider an LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx \quad (1)$$

described in standard notation by

$$G := \left[\begin{array}{c|c} A & B \\ \hline C & \mathbf{0} \end{array} \right]. \quad (2)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$, state vector $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathcal{U} \subset \mathbb{R}^p$ (with \mathcal{U} a compact set), and output $y(t) \in \mathbb{R}^m$.

Consider the following two definitions of reachable sets.

Definition 1: Given a target (unsafe) set of states $\mathcal{X}_f \subset \mathbb{R}^n$ and the time interval $\tau \in [t, t_f]$, the backward reachable set of system (2) at time t is defined as $\mathcal{X}_t := \text{Reach}(\mathcal{X}_f)$, $\mathcal{X}_t \subseteq \mathbb{R}^n$ and is the set of all states for which there exists a trajectory $x(\tau)$ such that $x(t_f) \in \mathcal{X}_f$ for all control input $u(\tau) \in \mathcal{U}$.

Definition 2: Given a target (unsafe) set of states $\mathcal{X}_f \subset \mathbb{R}^n$ and the time interval $\tau \in [t, t_f]$, the backward reachable set of the system $\dot{x} = Ax + Bu + Ld$, $L \in \mathbb{R}^{n \times q}$, $d \in \mathcal{D} \subset \mathbb{R}^q$, at time t is defined as $\mathcal{X}_t := \text{Reach}(\mathcal{X}_f)$, $\mathcal{X}_t \subseteq \mathbb{R}^n$ and is the set of all states for which there exists a trajectory $x(\tau)$ and a disturbance signal $d(\tau) \in \mathcal{D}$ such that $x(t_f) \in \mathcal{X}_f$ for all control input $u(\tau) \in \mathcal{U}$.

Now consider the following definitions.

Definition 3: The LTI system that consists of two subsystems

$$\dot{x}_1 = A_1 x_1 + \Delta_c x_2 \quad (3)$$

$$\dot{x}_2 = A_2 x_2 \quad (4)$$

with $A_1 \in \mathbb{R}^{k \times k}$, $A_2 \in \mathbb{R}^{(n-k) \times (n-k)}$, $\Delta_c \in \mathbb{R}^{k \times (n-k)}$, $x_1(t) \in \mathbb{R}^k$, and $x_2(t) \in \mathbb{R}^{(n-k)}$, is said to be *unidirectionally coupled* since the trajectories of (3) are affected by those of (4), while (4) evolves independently from (3).

Definition 4: Let there be a non-singular transformation matrix $T \in \mathbb{R}^{n \times n}$, such that $[z_1, z_2]^T = T^{-1}[x_1, x_2]^T$, and

$$\dot{z}_1 = A_1 z_1 + \tilde{\Delta}_c z_2 \quad (5)$$

$$\dot{z}_2 = A_2 z_2. \quad (6)$$

Then (5) and (6) are said to be *unidirectionally weakly-coupled* (in comparison to (3) and (4)) if

$$\|\tilde{\Delta}_c\|_\infty \leq \|\Delta_c\|_\infty, \quad (7)$$

where $\|\cdot\|_\infty$ denotes infinity norm.

Next, consider the following two lemmas which will be used in Section III.

Lemma 1: The Sylvester equation

$$EX + XF + H = \mathbf{0}, \quad (8)$$

with $E \in \mathbb{R}^{k \times k}$, $F \in \mathbb{R}^{m \times m}$, and $H \in \mathbb{R}^{k \times m}$, has a unique solution $X \in \mathbb{R}^{k \times m}$ if and only if the eigenvalue sum $\lambda_i(E) + \lambda_j(F) \neq 0$, $\forall i \in \{1, \dots, k\}$ and $\forall j \in \{1, \dots, m\}$.

Proof: cf. [17, Lem. 2.7]. ■

Let us now introduce the Schur form of a matrix.

Lemma 2: For any real matrix $M \in \mathbb{R}^{n \times n}$, there exists an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that $U^T M U = \tilde{M}$ is upper (quasi) triangular, and the eigenvalues of M are the eigenvalues of the block diagonals (each of dimension 2 or less) of \tilde{M} . Furthermore, the matrix U can be chosen to order the eigenvalues arbitrarily.

Proof: cf. [18, Thm's 7.1.3 and 7.4.1] and [19, 5R]. ■

Remark 1: It is easy to see that there always exists a partitioning of \tilde{M} such that $\tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \mathbf{0} & \tilde{M}_{22} \end{bmatrix}$.

Finally, a linear transformation of a set $\mathcal{X} \subseteq \mathbb{R}^n$ using an invertible transformation matrix $T \in \mathbb{R}^{n \times n}$ is $\mathcal{V} := \{v \in \mathbb{R}^n \mid v = T^{-1}x, x \in \mathcal{X}\}$. This, with an abuse of notation, is sometimes stated as $\mathcal{V} = T^{-1}\mathcal{X}$.

III. METHODOLOGY

Applying the results of Lemma 2 as in [20], we obtain an upper triangular A matrix for (2). We then perform a second similarity transformation and obtain a decoupled (or weakly-coupled) block diagonal matrix by solving a Sylvester equation (or an optimization problem). Therefore, we effectively decompose the system into two either completely decoupled or unidirectionally weakly-coupled subsystems.

In the case where the decomposition is decoupled, the reachable set in Definition 1 is computed separately for each isolated subsystem. When the decomposed subsystems are unidirectionally weakly-coupled, the reachability problem in Definition 1 is solved for the independent subsystem, whereas for the remaining subsystem, the effect of coupling is accounted for by treating the coupled terms as disturbance inputs and solving the reachability problem in Definition 2. For both decoupled and unidirectionally weakly coupled decompositions, the intersection of back projections of the lower dimensional reachable sets is an overapproximation of the actual reachable set in the transformed coordinate space.

When the control input across the decomposed subsystems is non-disjoint, a constrained optimization problem is solved in order to make one of the subsystems uncontrollable.

In the following analysis, we assume a partitioning of (2) that results in exactly two subsystems. However, the proposed method is generalizable to N subsystems by applying the same decomposition algorithm to each subsystem iteratively. A higher number of subsystems (i.e. iterated decomposition) may result in a more conservative overapproximation of the actual reachable set.

We now apply Lemma 2 with transformation matrix $U \in \mathbb{R}^{n \times n}$ to (2) to obtain

$$\tilde{G} = \left[\begin{array}{c|c} U^T A U & U^T B \\ \hline C U & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{C}_2 & \mathbf{0} \end{array} \right] \quad (9)$$

with $\tilde{A}_{11} \in \mathbb{R}^{k \times k}$, $\tilde{A}_{12} \in \mathbb{R}^{k \times (n-k)}$, $\tilde{A}_{22} \in \mathbb{R}^{(n-k) \times (n-k)}$, $\tilde{B}_1 \in \mathbb{R}^{k \times p}$, $\tilde{B}_2 \in \mathbb{R}^{(n-k) \times p}$, $\tilde{C}_1 \in \mathbb{R}^{m \times k}$, and $\tilde{C}_2 \in \mathbb{R}^{m \times (n-k)}$.

A. Disjoint Control Input

Consider the case in which the subsystems have independent control inputs. In other words, the portion of the control input that influences the i^{th} subsystem is

$$u_i \in \mathcal{U}_i \subset \mathbb{R}^{p_i}, \quad p = \sum_{i=1}^N p_i, \quad (10)$$

where $N = 2$ is the number of subsystems.

Proposition 1: If a solution $X \in \mathbb{R}^{k \times (n-k)}$ to the Sylvester equation

$$\tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12} = \mathbf{0} \quad (11)$$

exists, then a transformation

$$W = \left[\begin{array}{cc} I_{k \times k} & X \\ \mathbf{0} & I_{(n-k) \times (n-k)} \end{array} \right] \in \mathbb{R}^{n \times n}, \quad (12)$$

makes (9) completely decoupled.

Proof: Applying the transformation W to \tilde{G} , we obtain

$$\begin{aligned} & \left[\begin{array}{c|c} W^{-1} \tilde{A} W & W^{-1} \tilde{B} \\ \hline \tilde{C} W & \mathbf{0} \end{array} \right] \\ &= \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12} & B_{d1} \\ \mathbf{0} & \tilde{A}_{22} & B_{d2} \\ \hline C_{d1} & C_{d2} & \mathbf{0} \end{array} \right] \quad (13) \\ &= \left[\begin{array}{cc|c} \tilde{A}_{11} & \mathbf{0} & B_{d1} \\ \mathbf{0} & \tilde{A}_{22} & B_{d2} \\ \hline C_{d1} & C_{d2} & \mathbf{0} \end{array} \right] := G_d. \end{aligned}$$

Notice that

$$G_d = \left[\begin{array}{c|c} \tilde{A}_{11} & B_{d1} \\ \hline C_{d1} & \mathbf{0} \end{array} \right] + \left[\begin{array}{c|c} \tilde{A}_{22} & B_{d2} \\ \hline C_{d2} & \mathbf{0} \end{array} \right] = G_{d1} + G_{d2}. \quad (14)$$

Therefore, we have effectively decoupled system (2) through a similarity transformation to a new coordinate system $z = T^{-1}x$, $T^{-1} = W^{-1}U^T$. Reachability analysis (in this transformed coordinate space) can then be performed on each lower-dimensional subsystem G_{d1} and G_{d2} separately.

Now consider the case in which there is no solution to the Sylvester equation (11).

Proposition 2: If (11) does not have a solution, then the transformation (12), with

$$X = \arg \min \|\tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12}\|_{\infty} \quad (15)$$

results in unidirectionally weakly-coupled subsystems.

Proof: Consider $A_c := \tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12} \neq \mathbf{0}$ in (13). In the transformed coordinate state space, (1) becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & A_c \\ \mathbf{0} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix} u, \quad (16)$$

with $z = (UW)^{-1}x$. It is clear that $z_2 \in \mathbb{R}^{(n-k)}$ evolves independently of $z_1 \in \mathbb{R}^k$ since

$$\dot{z}_2 = \tilde{A}_{22} z_2 + B_{d2} u_2. \quad (17)$$

However, z_1 is affected by z_2 through A_c . That is, we have

$$\dot{z}_1 = \tilde{A}_{11} z_1 + B_{d1} u_1 + A_c z_2. \quad (18)$$

Note that u_i , $i = \{1, 2\}$ is the effective portion of the input vector u for the i^{th} subsystem. Minimization of the ∞ -norm of A_c therefore translates into minimizing (i.e. *weakening*) the worst-case unidirectional coupling of z_1 with z_2 . To see this, let $X^* = \arg \min \|\tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12}\|_{\infty}$. Then the hypothesis $\|\tilde{A}_{12}\|_{\infty} < \|\tilde{A}_{11} X^* - X^* \tilde{A}_{22} + \tilde{A}_{12}\|_{\infty}$ would imply that $X^* = \mathbf{0}$ can never be a solution. Since there are no constraints in (15) imposing this restriction, by contradiction we conclude that

$$\|\tilde{A}_{11} X^* - X^* \tilde{A}_{22} + \tilde{A}_{12}\|_{\infty} \leq \|\tilde{A}_{12}\|_{\infty}. \quad (19)$$

Therefore, according to Definition 4, the resulting subsystems (17) and (18) are unidirectionally weakly-coupled. ■

Remark 2: The objective function of (15) is convex, and therefore, a solution always exists.

Reachability is first performed on the isolated subsystem (17) according to Definition 1. The maximum value of the corresponding reachable set in the worst case direction ($\|z_2\|_{\infty}$) is recorded. For subsystem (18), the coupling term $A_c z_2$ is treated as disturbance. Using the multiplicative property of induced ∞ -norm we have

$$\|A_c z_2\|_{\infty} \leq \|A_c\|_{\infty} \cdot \|z_2\|_{\infty}. \quad (20)$$

Therefore an upper bound for the disturbance to this subsystem is obtained and a conservative overapproximation of the reachable set, defined in Definition 2, in the corresponding lower-dimensional subspace is computed.

B. Non-Disjoint Control Input

Now consider a decomposition in which the same control input affects both subsystems; that is, the control input is no longer disjoint. In this case, reachability analysis cannot be completed on the two subsystems independently since they are both coupled to each other through the input. For example, consider the case in which a control value deemed optimal for one subsystem is in fact non-optimal for the other subsystem.

One way to remedy this issue is by ensuring that at least one of the subsystems in the transformed coordinate space has null input matrix, i.e. $\exists i \in \{1, 2\}$ s.t. $B_{di} = \mathbf{0}$. We refer to one such subsystem as *trivially-uncontrollable*.

It is clear that in such a case the (otherwise non-disjoint) control action does not affect the evolution of the reachable set of the trivially-uncontrollable subsystem. Therefore, an

optimal control input for the subsystem with nonzero input matrix is also optimal for the full-order system.

More formally, if either the pair (\tilde{A}_{22}, B_{d2}) or the pair (\tilde{A}_{11}, B_{d1}) in (13) is made trivially-uncontrollable, reachability analysis can be performed as in the disjoint control input case, separately for each subsystem.

Proposition 3: The transformation (12), with

$$X = \arg \min \|\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12}\|_{\infty} \quad (21)$$

$$\text{s.t. } X\tilde{B}_2 = \tilde{B}_1$$

results in unidirectionally coupled subsystems. Furthermore, the pair (\tilde{A}_{11}, B_{d1}) is trivially-uncontrollable.

Proof: Consider $B_d = W^{-1}\tilde{B}$ in (13). We have,

$$\begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix} = \begin{bmatrix} I & -X \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 - X\tilde{B}_2 \\ \tilde{B}_2 \end{bmatrix}. \quad (22)$$

The equality constraint in (21) simply enforces $B_{d1} = \mathbf{0}$. ■

The resulting subsystems can now be treated as in the disjoint control input case, and hence an overapproximation of the reachable set in each subspace can be computed.

Remark 3: The ∞ -norm of the unidirectional coupling term A_c obtained through Proposition 3 may no longer be less than that of \tilde{A}_{12} . This, in addition to the fact that the upper subsystem is made trivially-uncontrollable, may lead to an overly conservative reachable set computation. However, the flexibility of the Schur form in placing the eigenvalues in any order along the diagonal of \tilde{A} can be exploited to make this subsystem evolve with slower dynamics, which could potentially prevent the excessive overapproximation.

For both disjoint and non-disjoint control cases, the overapproximation of the actual reachable set of the full-order system in \mathbb{R}^n can be obtained using the following corollary.

Corollary 1 (cf. [14]): Let \mathcal{Z}_t^{di} , $i = \{1, 2\}$, be the computed lower-dimensional overapproximative reachable set of subsystem i . Then the transformation of the intersection of the back-projection of these sets onto \mathbb{R}^n overapproximates the actual full-order reachable set \mathcal{X}_t of system (2). That is,

$$\hat{\mathcal{X}}_t := T((\mathcal{Z}_t^{d1} \times \mathcal{R}_2) \cap (\mathcal{Z}_t^{d2} \times \mathcal{R}_1)) \supseteq \mathcal{X}_t, \quad (23)$$

where $T = UW$ is the transformation matrix and \mathcal{R}_i , $i = \{1, 2\}$, is the appropriately dimensioned subspace of the i^{th} subsystem.

C. Extension to Hybrid Systems

The extension of our algorithm to hybrid dynamical systems is fairly straight forward. Consider the hybrid automaton $(Q, \mathfrak{X}, f, \mathcal{U}, \Sigma, R)$, with discrete modes $Q = \{q_i\}$, continuous states $x \in \mathfrak{X}$, control inputs $u \in \mathcal{U}$, control actions $\sigma \in \Sigma$, vector field $f : Q \times \mathfrak{X} \times \mathcal{U} \rightarrow \mathfrak{X}$, $f : (q_i, x, u) \mapsto A_i x + B_i u$, and transition function $R : Q \times \mathfrak{X} \times \mathcal{U} \times \Sigma \rightarrow Q \times \mathfrak{X}$.

Let $\mathcal{X}_f(q_i)$ (a set of continuous states in mode q_i) be the target set and $\mathcal{W}(q_i)$ the reachable set. Also, let T_i be the transformation matrix for mode q_i obtained from the complexity reduction algorithm in Section III. As in [21],

reachability calculations proceed in each mode in parallel such that for mode q_i the reach-avoid operation becomes

$$T_i \text{Reach}(T_i^{-1} \mathcal{X}_f(q_i), T_i^{-1} \mathcal{W}(q_i)). \quad (24)$$

In case of a switched system with two modes q_i and q_j and an identity reset map, the backward reachable set \mathcal{X}_t can be directly calculated as

$$\mathcal{X}_t = T_j \text{Reach}\left(q_j, T_j^{-1} T_i \text{Reach}(q_i, T_i^{-1} \mathcal{X}_f(q_i))\right) \quad (25)$$

where T_i and T_j are the transformation matrices for modes q_i and q_j respectively. Reachability analysis is then performed on lower-dimensional subsystems in each mode.

IV. NUMERICAL EXAMPLES

Although complexity reduction through Schur-based decomposition can be used in conjunction with any reachability technique, we demonstrate the applicability and practicality of our method using three examples that employ the Level Set Toolbox (LS) [22]. While LS has mainly been used for systems of dimension 5 or less [23], our complexity reduction approach can facilitate the use of LS for a class of higher dimensional systems for which safety controller synthesis and handling of non-convex or arbitrarily-shaped sets is important.

All computations in the following are performed on a dual core Intel-based computer with 2.8 GHz CPU, 6 MB of cache and 3 GB of RAM running MATLAB 7.5.

A. Arbitrary 3D System

Consider an arbitrary 3D LTI system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} -0.5672 & -0.7588 & -0.6282 \\ 3.1364 & -1.1705 & 2.3247 \\ 1.8134 & -1.7689 & -2.6930 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0731 & -0.1639 \\ -0.7377 & -0.3578 \\ 0.1470 & 0.2410 \end{bmatrix}$$

and $u = [u_1, u_2]^T \in \mathbb{R}^2$, $\|u\|_{\infty} \leq 1$. We choose a target (unsafe) set $\mathcal{X}_f \subset \mathbb{R}^3$ such that in the transformed coordinate space, $\mathcal{Z}_f = \{z \in \mathbb{R}^3 \mid \|z\|_{\infty} \leq 0.2, z = T^{-1}x, x \in \mathcal{X}_f\}$ where T is the transformation matrix obtained through our algorithm.

We decompose this system, using Proposition 1, into two subsystems (one 2D and one 1D) with disjoint control for each subsystem. The matrices A and B in the transformed coordinate space are

$$A_d = \begin{bmatrix} -1.6653 & -3.4560 & 0 \\ 1.8706 & -1.4653 & 0 \\ 0 & 0 & -1.3000 \end{bmatrix}, \quad B_d = \begin{bmatrix} -0.7530 & 0 \\ 0.0640 & 0 \\ 0 & 0.2500 \end{bmatrix}.$$

Hence the decoupled subsystems are

$$\dot{z}_1 = \begin{bmatrix} -1.6653 & -3.4560 \\ 1.8706 & -1.4653 \end{bmatrix} z_1 + \begin{bmatrix} -0.7530 \\ 0.0640 \end{bmatrix} u_1$$

$$\dot{z}_2 = [-1.3000] z_2 + [0.2500] u_2.$$

We obtain an overapproximation of the actual reachable set, as shown in Fig. 1. The reachability calculation is performed over a grid with 101 nodes in each dimension for $t_f = 2$ seconds. The computation time for the actual and the Schur-based reachable sets (including decomposition and projections) were 5823.73 and 22.87 seconds, respectively—a significant reduction.

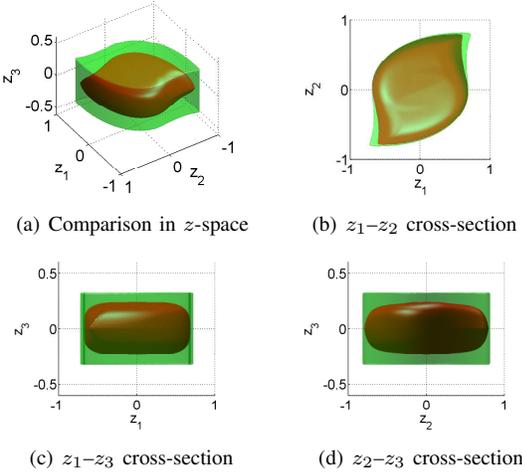


Fig. 1. Schur-based overapproximation (transparent light) vs. actual (solid dark) reach sets in the transformed coordinate space for example A (3D)

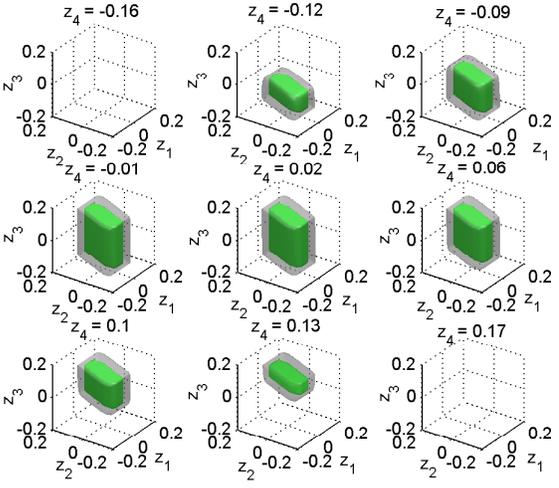


Fig. 2. Schur-based (solid dark) vs. actual (transparent light) safe sets in the transformed coordinate space for example B (4D)

B. 4D Aircraft Dynamics

Consider longitudinal aircraft dynamics $\dot{x} = Ax + B\delta_e$,

$$A = \begin{bmatrix} -0.0030 & 0.0390 & 0 & -0.3220 \\ -0.0650 & -0.3190 & 7.7400 & 0 \\ 0.0200 & -0.1010 & -0.4290 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.0100 \\ -0.1800 \\ -1.1600 \\ 0 \end{bmatrix}$$

with state $x = [u, \alpha, \dot{\theta}, \theta]^T \in \mathbb{R}^4$ comprised of deviations in aircraft speed, angle of attack, pitch-rate, and pitch angle respectively, and with input $\delta_e \in [-13.3^\circ, 13.3^\circ] \in \mathbb{R}$ the elevator deflection. These matrices represent stability derivatives of a Boeing 747 aircraft cruising at an altitude of 40 kft with speed 774 ft/sec [24]. We define a target (unsafe) set \mathcal{X}_f such that in the transformed coordinate space $\mathcal{Z}_f = \{z \in \mathbb{R}^4 \mid \|z\|_\infty > 0.15, z = T^{-1}x, x \in \mathcal{X}_f\}$ where T is the transformation matrix obtained through our algorithm.

We first decompose the system into two 2D subsystems. Since the control input is non-disjoint across the resulting subsystems, we use Proposition 3 and obtain unidirectionally coupled subsystems, one of which is trivially-uncontrollable. The reachability calculation is performed over a grid with 41 nodes in each dimension for $t_f = 5$ seconds. The computation time for the actual and the Schur-based reachable sets (including decomposition and projections) were 28546.8 and 54.64 seconds, respectively.

Since the computed sets are 4D, we plot a series of 3D snapshots of these 4D objects at specific values of z_4 (Fig. 2). The aircraft flight envelope (safe) is represented by the area inside the shaded regions.

C. 8D Distillation Column

Finally, consider the dynamic model of a binary distillation column $\dot{x} = Ax + Bu$ obtained from [25] with A and B given in (26). The input $u = [u_1, u_2]^T \in \mathbb{R}^2$ with $u_1, u_2 \in [0, 1]$ is comprised of reflux flow and boilup flow, respectively.

The full-order system with state vector $x \in \mathbb{R}^8$ is first decomposed into two (unidirectionally coupled) 4D subsystems using Proposition 3, since the control vector is

non-disjoint across the two candidate subsystems. Similarly, each of these 4D subsystems is decomposed into two 2D subsystems. Since the upper 4D subsystem is made trivially-uncontrollable through (21), its decomposition is disjoint and therefore Proposition 1 is used to obtain the 1st and 2nd (decoupled) 2D subsystems. On the other hand, for the lower 4D subsystem the decomposition results in non-disjoint control input. Therefore Proposition 3 is employed and the 3rd and 4th (unidirectionally coupled) 2D subsystems are obtained.

Reachability is first performed on the 3rd and 4th subsystems while taking the effect of unidirectional coupling into account. Next, the reachable sets of the 1st and 2nd subsystems are computed while treating the effect of the 3rd and 4th subsystems as disturbance.

We label the 2D transformed state subspaces as $\tilde{w}_1 = [w_1, w_2]^T$, $\tilde{w}_2 = [w_3, w_4]^T$, $\tilde{q}_1 = [q_1, q_2]^T$, and $\tilde{q}_2 = [q_3, q_4]^T$. Notice that $\mathbb{R}^4 \ni q = [\tilde{q}_1, \tilde{q}_2]^T = T_3^{-1}\tilde{z}_2$, $\mathbb{R}^4 \ni w = [\tilde{w}_1, \tilde{w}_2]^T = T_2^{-1}\tilde{z}_1$, and $\mathbb{R}^8 \ni z = [\tilde{z}_1, \tilde{z}_2]^T = T_1^{-1}x$ with $\tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^4$.

As in previous examples, for simplicity of calculations, we assume that the target (unsafe) set $\mathcal{X}_f \subset \mathbb{R}^8$ is chosen such that the transformations $T_1^{-1} \in \mathbb{R}^{8 \times 8}$, $T_2^{-1} \in \mathbb{R}^{4 \times 4}$, and $T_3^{-1} \in \mathbb{R}^{4 \times 4}$ result in $\mathcal{W}_f := \{w \in \mathbb{R}^4 \mid \|w\|_\infty > 20\}$ and $\mathcal{Q}_f := \{q \in \mathbb{R}^4 \mid \|q\|_\infty > 20\}$. The target sets for the 2D subsystems is simply the projection of \mathcal{W}_f and \mathcal{Q}_f onto their corresponding subspaces.

Lower dimensional reachability is performed over a grid with 101 nodes in each dimension for $t_f = 6$ seconds. The overall computation time (including decomposition and projections) was 94.31 seconds. The complement of the shaded regions in Fig. 3 overapproximate the reachable (unsafe) set in each of the 2D subspaces. The full 8D reachable set is the intersection of the back-projection of the 2D reach sets.

The actual reachable set is not shown since it is prohibitively computationally expensive to compute with LS.

$$A = \begin{bmatrix} -0.5774 & 3.0567 & 0.0073 & -0.8121 & 0.3034 & -0.3035 & 0.0072 & -0.1542 \\ -2.7290 & -0.7147 & -0.3430 & 1.5321 & 0.6643 & 0.2896 & -0.0013 & 0.0926 \\ 0 & 0 & -0.3891 & -0.9956 & 0.0182 & 0.0235 & 0.0049 & 0.0506 \\ 0 & 0 & 1.3640 & -1.3363 & -0.9037 & -0.4686 & -0.0009 & -0.1887 \\ 0 & 0 & 0 & 0 & -0.7357 & -0.2275 & -0.0082 & -0.0021 \\ 0 & 0 & 0 & 0 & 0 & -0.2259 & 0.0021 & -0.0457 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0052 & 0.0024 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0755 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0335 & -0.1228 \\ -0.4534 & -0.0711 \\ -0.8005 & -0.2612 \\ 0.5497 & -0.1344 \\ 1.2886 & -0.0504 \\ 0.3132 & -0.2249 \\ 0.7117 & -0.6994 \\ 0.0599 & -0.3014 \end{bmatrix} \quad (26)$$

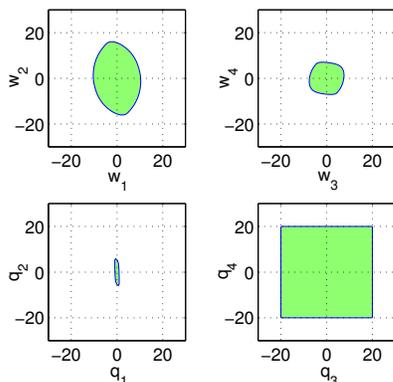


Fig. 3. Safe set of example C (8D) in transformed 2D subspaces

V. CONCLUSIONS AND FUTURE WORKS

This paper presented a Schur-based decomposition for reachability analysis of LTI systems. This decomposition has considerable potential for reducing the computational complexity in reachable set calculations, especially for reachability tools that are computationally intensive (such as the Level Set Toolbox). The decomposition was evaluated in terms of whether the resulting subsystems had disjoint or non-disjoint control inputs. In the event that a Sylvester equation can be solved, the decomposition yields two decoupled subsystems. When the Sylvester equation cannot be solved, its minimization yields two weakly coupled subsystems. A constrained optimization problem is considered for the case in which the control input is non-disjoint across decomposed subsystems. We applied this technique to a variety of examples computed with the Level Set Toolbox, and found computational time significantly reduced when our method was employed.

In future work, we plan to apply this technique to the problem of safety verification of an automatic drug delivery system for anesthesia. We require not only reachable set calculation for an irregularly-shaped, non-convex target set, but also safety controller synthesis, for a hybrid system with 6-dimensional LTI continuous dynamics. The complexity reduction of the technique proposed here will enable the use of the Level Set Toolbox for this system.

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