Stability of switched block upper-triangular linear systems with bounded switching delay: Application to large distributed systems

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Abstract— **With the advent of inexpensive and reliable wireless communications, distributed systems under cooperative control will become widespread. We consider distributed systems with switched linear subsystem dynamics, and focus on the stability of block upper-triangular switched linear systems with bounded switching delay, when switching between stable modes. Proving globally uniformly asymptotic stability (GUAS) of a switched block upper-triangular linear system can be reduced to proving GUAS for each of its block diagonal subsystems. We derive a scalable LMI-based test for GUAS under arbitrary switching. We also derive a partition of the state-space into regions in which switching will preserve GUAS despite a delay between the state measurements and switching time, when a global quadratic Lyapunov function does not exist. The resulting partitions asymptotically approach the standard piecewise quadratic Lyapunov based partitioning for stable switching. We evaluate our method on a formation of 100 vehicles under supervisory discrete control, and a switched linear system under remote control.**

I. Introduction

We consider stability of switched linear systems [1], [2], [3] with application to distributed control, motivated by problems in large decentralized systems such as military battle systems with heterogeneous autonomous fighters, formation flight in uncertain environments, or distributed power systems. Many systems under distributed control cannot be described through continuous state-space models alone – discrete modes may characterize distinct modes of operation, or may arise from the use of a hybrid controller (e.g., one which has both a fast response time and good noise rejection properties [4], a key property for many real-world systems). However, without efficient computational tools to analyze and control switched, distributed systems, the potential performance and robustness benefits of these advanced control techniques cannot be exploited.

We focus on proving GUAS of distributed systems under cooperative control in which each subsystem is a switched linear system. Distributed systems comprised of identical subsystems under cooperative control can be transformed to have a block upper-triangular structure [5], [6]. Recent work in vehicle formation control [5], [6], [7], [8], consensus and swarming [9], [10], [11], mobile sensor networks [12], [13], [14], control over uncertain channels [15], [16], [17], and optimal control [18], [19], all under topological constraints, assume that the dynamics of each subsystem can be represented by a continuous system. We extend techniques from [5], [6] to prove stability of distributed systems with switched linear subsystem dynamics.

While finding a global quadratic Lyapunov function (GQLF) to prove global uniform asymptotic stability (GUAS) under arbitrary switching, or finding a piecewise quadratic Lyapunov function (PQLF) to prove GUAS under switching constraints, can be accomplished by solving a set of Linear Matrix Inequalities (LMIs) [2], [20], for distributed systems (such as a formation of hundreds of vehicles), this approach may be computationally infeasible. Solving LMIs is typically fast and efficient, however for systems with very high dimension an unreasonable amount of memory may be required.

To preserve block upper-triangular structure, we assume that the continuous controllers are implemented in a distributed manner, whereas mode switches are dictated by a centralized, supervisory discrete controller. We show that proving GUAS under a given switching scheme for block upper-triangular systems is equivalent to proving GUAS for each of its block diagonal subsystems under this same switching scheme. In cases where a system is only GUAS under constrained switching, we synthesize state-based constraints for switching which guarantee GUAS despite a switching delay. This switching delay could model communication delays inherent to most distributed systems.

Our main contributions are 1) a scalable, computationally efficient test for GUAS under arbitrary switching of block upper-triangular systems, 2) a proof that for state constraint-based switching, only that subset of the state space corresponding to the block diagonal subsystems that are not GUAS under arbitrary switching need to be taken into consideration, 3) a method of synthesizing state constraints that guarantee GUAS despite a switching delay, and 4) a proof that these state constraints asymptotically approach standard (delay-free) Lyapunov based constraints.

In this paper, Section II presents the problem formation. In Section III, we show that a block upper-triangular switched linear system is GUAS under a given switching scheme if and only if each of its block diagonal subsystems is GUAS under that switching scheme as well. Section IV presents a method for synthesizing state constraints that guarantee GUAS despite a switching delay. Section V provides two examples: 1) cooperative control of a large vehicle formation under discrete supervisory control and 2) remote supervisory control of a switched linear system. Section VI provides conclusions and directions for future work.

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II. Problem Formulation

Consider a fleet of N identical vehicles in which the three position variables are decoupled, and the acceleration in each direction is controlled separately. We can thus limit our analysis without loss of generality to vehicles moving in one dimension. The i^{th} vehicle's dynamics are given by

$$\dot{\overline{x}}_i = A\overline{x}_i + Bu_i \tag{1}$$

with $\overline{x}_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}$, and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Furthermore assume a local full-state feedback controller is implemented on each vehicle such that the closed loop dynamics A_{cl} of the individual vehicles are stable.

The formation dynamics of a fleet of linear vehicles with dynamics (1) can be described by [5], [6]

$$\dot{\overline{x}} = (I_N \otimes A_{cl} + L \otimes BK)\overline{x} \tag{2}$$

for which I_N is the $N \times N$ identity matrix, $\otimes$ denotes the Kronecker product, $\overline{x} = [\overline{x}_1^T, \overline{x}_2^T, \cdots, \overline{x}_N^T]^T \in \mathbb{R}^{2N}$, $K \in \mathbb{R}^{1\times 2}$ is the linear formation feedback controller, identical for all vehicles, and $L \in \mathbb{R}^{N\times N}$ is the graph Laplacian describing the fixed communication topology of the formation. Consider the case in which a supervisory logic controller switches the linear formation feedback controller of all of the vehicles, such that $K = K_\sigma$, where σ is a piecewise constant switching signal.

Applying the transformation $T = U \otimes I_2$ to (2), with a Schur transformation matrix U as in [5], [6], such that $\widetilde{L} = U^{-1}LU$ is upper triangular and the diagonal entries of $\widetilde{L}$ are the eigenvalues of L, results in a block upper-triangular system in the transformed coordinates $x = T^{-1}\overline{x}$.

$$\dot{x} = (I_N \otimes A_{cl} + \widetilde{L} \otimes BK_\sigma)x \tag{3}$$

We assume that K_p has been chosen such that (7) is Hurwitz for all $p \in \mathcal{P}$ and all eigenvalues of L.

More generally, consider a switched linear system

$$\dot{x} = M_\sigma x \tag{4}$$

with $x \in \mathbb{R}^n$, $\sigma : [0, \infty) \to \mathcal{P} \subset \mathbb{N}$ a piecewise constant switching signal, and $\mathcal{M} := \{M_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$ a family of block upper-triangular Hurwitz state matrices indexed by p.

Definition 1: A family of block upper-triangular state matrices is denoted $\mathcal{M} := \{M_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P} \subset \mathbb{N}\}$, indexed by p. The state matrix

$$M_p = \begin{bmatrix} A_p^1 & X_{12} & \cdots & X_{1N} \\ 0 & A_p^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & X_{(N-1)N} \\ 0 & 0 & \cdots & A_p^N \end{bmatrix} \tag{5}$$

consists of square blocks $A_p^i \in \mathbb{R}^{n_i\times n_i}$, $\sum_{i=1}^{N} n_i = n$, $i \in \{1, ..., N\}$, and non-zero, off-diagonal elements X_{ij} of appropriate dimension. For $\mathcal{A}^i := \{A_p^i \in \mathbb{R}^{n_i\times n_i} : p \in \mathcal{P}\}$, $x_i \in \mathbb{R}^{n_i}$ the corresponding subset of the state vector $x \in \mathbb{R}^n$, and $\sigma(\cdot) : [0, \infty) \to \mathcal{P}$ a piecewise constant switching signal,

$$\dot{x}_i = A_\sigma^i x_i \tag{6}$$

is the i^{th} *block diagonal subsystem* of the switched linear system (4).

For the vehicle formation (3), the block diagonal subsystems (6) are

$$\dot{x}_i = (A_{cl} + \lambda_i BK_\sigma)x_i \tag{7}$$

with λ_i an eigenvalue of L.

Definition 2: (From [2].) The system (4) is *globally uniformly asymptotically stable (GUAS) under* Σ^*, a set of piecewise constant switching signals, if there exist constants $c, \mu > 0$ such that the solution $x(t) = \Phi_\sigma(t, 0)x(0)$ with state transition matrix $\Phi_\sigma(t, 0)$ satisfies the equivalent conditions

$$\begin{aligned} \|x(t)\| &\leq ce^{-\mu t}\|x(0)\| \\ \|\Phi_\sigma(t, 0)\| &\leq ce^{-\mu t} \end{aligned} \tag{8}$$

for all $t \geq 0$, any initial state $x(0)$, and *any switching signal* $\sigma(\cdot) \in \Sigma^*$.

Remark 1: For $\Sigma^* = \{p\}$, $p \in \mathcal{P}$, (i.e. $\sigma(t) \equiv p$), Definition 2 is equivalent to GUAS of a linear system.

Remark 2: For Σ^* the set of all piecewise constant switching signals, (4) is *GUAS under arbitrary switching*.

For distributed systems under cooperative control, block diagonal subsystems (6) have the same number of states as the individual subsystems. Thus, an approach for proving the stability of (4) by solely analyzing its block diagonal subsystems would be highly scalable, as its complexity would be linear in the number of subsystems.

In addition, communication and other types of delays are often present. Delays can arise from a remote supervisory discrete controller (such as a human operator triggering mode changes) receiving delayed measurements, or the time required to synchronize a simultaneous mode switch amongst several subsystems. Consequently, we assume that there is a switching delay T_D between the state measurements and switching time – the discrete controller will only have access to a delayed state measurement $x(\tau - T_D)$. If (4) is not GUAS under arbitrary switching, it is of interest to determine whether a switching scheme robust to a delay between state measurements and switching time, or a *switching delay*, can be developed.

Problem 1: Given a switched linear system (4), synthesize a state constraint-based switching scheme which preserves GUAS despite a switching delay, by analyzing the stability of its block diagonal subsystems (6).

Sub-problem 1: Show that a switched linear system (4) is GUAS for a given set of switching signals by analyzing the stability of its block diagonal subsystems (6).

In solving these problems, we obtain a scalable method of guaranteeing GUAS of a switched linear system (4) in a manner that is robust to a switching delay. We begin by addressing Sub-problem 1, and then use these results to solve Problem 1.

III. Stability under Arbitrary Switching

A switched linear system (4) with $\mathcal{M}$ a family of Hurwitz upper-triangular state matrices is GUAS under arbitrary switching [2], [21], [22], [23]. We extend these results to the case where $\mathcal{M}$ is a family of Hurwitz block upper-triangular systems, and prove that (4) is GUAS under a given switching scheme if and only if each block diagonal subsystem of (4) is also GUAS under this switching scheme.

With $||H|| := \max_{x\neq 0} \frac{||Hx||}{||x||}$ for $H \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, we have $||Hx|| \leq ||H||||x||$. Recall that for a linear system $\dot{x} = Ax+Bu$ with Hurwitz matrix A, the system is GUAS if the input u is exponentially decaying, i.e. there exist positive constants c, μ satisfying $||u(t)|| \leq ce^{-\mu t}||u(0)||$.

Theorem 1: A switched linear system (4) is GUAS under a set of piecewise constant switching signals Σ^*, if and only if each block diagonal subsystem (6) is GUAS under Σ^*.

Proof: Assume without loss of generality that

$$M_p = \begin{bmatrix} A_p^1 & B_p \\ 0 & A_p^2 \end{bmatrix} \tag{9}$$

with $A_p^i \in \mathbb{R}^{n_i\times n_i}$, $n_1+n_2 = n$, $B_p \in \mathbb{R}^{n_1\times n_2}$, and $x = [x_1^T, x_2^T]^T$, with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$.

If: Assume $\dot{x}_1 = A_\sigma^1 x_1$ and $\dot{x}_2 = A_\sigma^2 x_2$ GUAS under Σ^*, then from Definition 2, $\|x_2(t)\| \leq c_2e^{-\mu_2 t}\|x_2(0)\|$, $\forall\sigma(\cdot) \in \Sigma^*$ for some c_2, $\mu_2 > 0$. Treating x_2 as an exponentially decaying input to x_1,

$$x_1(t) = \Phi_\sigma^1(t,0)x_1(0) + \int_0^t \Phi_\sigma^1(t,\tau)B_{\sigma(\tau)}x_2(\tau)d\tau \tag{10}$$

with $\|\Phi_\sigma^1(t,\tau)\| \leq ae^{-\mu(t-\tau)}$, $\forall\sigma(\cdot) \in \Sigma^*$, for a, $\mu > 0$, from Definition 2. Since $\|B_{\sigma(\tau)}\| \leq \max_{p\in\mathcal{P}} \|B_p\| := \|B_{\max}\|$,

$$\begin{aligned} \|x_1(t)\| &\leq \|\Phi_\sigma^1(t,0)\|\cdot\|x_1(0)\| \\ &\quad + \|B_{\max}\| \int_0^t \|\Phi_\sigma^1(t,\tau)\|\cdot\|x_2(\tau)\|d\tau \\ &\leq c_1e^{-\mu_1 t}\|x_1(0)\|, \ \forall\sigma(\cdot) \in \Sigma^* \end{aligned} \tag{11}$$

for c_1, $\mu_1 > 0$, hence (4) is GUAS under Σ^*.

Only if: Assume (4) is GUAS under Σ^*, then by Definition 2, there exist $c, \mu > 0$ such that $\|x(t)\| \leq ce^{-\mu t}$. This holds for $x(t)$ if and only if it holds for all subsets $x_i(t)$ of $x(t)$. If there do not exist positive constants c_i,μ_i, satisfying $\|x_i(t)\| \leq c_ie^{-\mu_i t}\|x_i(0)\|$ $\forall\sigma(\cdot) \in \Sigma^*$ for $i = \{1, 2\}$, then c, μ do not exist, resulting in a contradiction.

An extension to N block upper-triangular matrices of arbitrary dimension proceeds by induction, beginning with the bottom block diagonal subsystem. ■

Corollary 1: A switched linear system (4) is GUAS under arbitrary switching if and only if each block diagonal subsystem (6) is GUAS under arbitrary switching.

Consider a P mode, N block system (4), with each subsystem (6) of dimension n. Analysis of (4) as a whole would involve solving $P+1$ LMIs in $\mathbb{R}^{Nn\times Nn}$ – for large N this quickly becomes prohibitively expensive in terms of memory requirements. However, applying Corollary 1, we solve N sets of $(P+1)$ LMIs in $\mathbb{R}^{n\times n}$, each easily computed. Figure 1 depicts the number of decision variables Matlab requires to solve an LMI to prove GUAS under arbitrary switching for a full system, as opposed to for an individual subsystem. Clearly, lowering the dimension of the matrices involved has a significant impact on the number of decision variables required to solve the LMI.

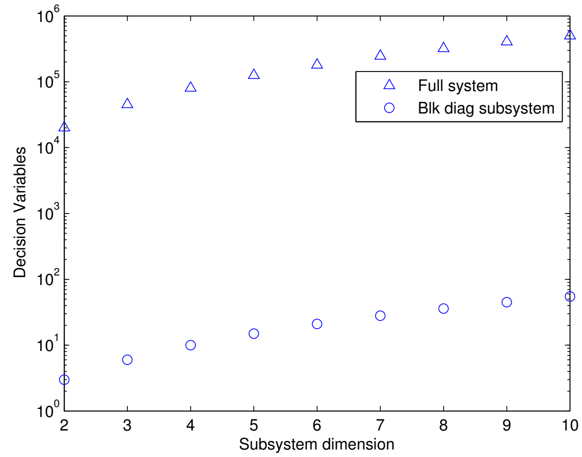

Fig. 1. Decision variables required in Matlab to solve an LMI to prove GUAS under arbitrary switching for a) a two mode distributed system with 100 subsystems (△) of total dimension $100n$, and b) a single block diagonal two mode subsystem (○) of dimension n, for $n \in \{2, \cdots, 10\}$.

The derivation of Theorem 1 and Corollary 1 hinges on three key assumptions: (1) a fixed communication topology, (2) all vehicles are identical at all times and (3) all vehicles have linear dynamics. These assumptions are required to preserve the properties of Kronecker multiplcation so that the system can be transformed into block upper-triangular form. If the communication topology changes, then the graph Laplacian will as well, requiring a new coordinate transformation to apply Theorem 3. However, if the communication topology is fairly reliable, this should not cause instability – so long as topology changes do not occur too quickly, a dwell time argument [24] can be used to show that this will not destabilize the system. Once the new communication framework has been established, our results can once again be applied to prove GUAS under arbitrary switching.

IV. Stability under State Constrained Switching with Switching Delay

Suppose that no GQLF can be found for one or more of the block diagonal subsystems. By Theorem 1, if a switching scheme can be generated such that these block diagonal subsystems (6) are GUAS under *constrained* switching, then (4) will be GUAS under constrained switching as well. Hence, only that subset of the state-space corresponding to the block diagonal subsystems that are not GUAS under arbitrary switching need to be taken into consideration, potentially simplifying implementation and relaxing switching restrictions as compared what would be required from analysis of the entire system. However, in contrast to Section III, the switching delay must be explicitly accounted for.

Consider the i^{th} block diagonal subsystem (6), and for ease of notation, omit the i (sub)superscript. For each mode $p \in \mathcal{P}$, let $V_p(x) = x^T P_p x$ be the associated Lyapunov function, with $x \in \mathbb{R}^n$ and $P_p = P_p^T > 0$ is real valued. Additionally define $Q_p(q) := -(A_q^T P_p + P_p A_q)$ to track the evolution of the Lyapunov function in mode q while $\sigma(t) = p$. Note that $Q_p(q) = Q_p^T(q)$ for all $p, q \in \mathcal{P}$.

Theorem 2: Consider a delay free switched system (6). Let Σ^* be the set of all piecewise constant switching signals $\sigma(\cdot) : [0, \infty) \to \mathcal{P}$ for which

$$V_{\sigma(\tau^-)}(x) - V_{\sigma(\tau)}(x) > 0 \tag{12}$$

for each switching time τ. Then (6) is GUAS under Σ^*.

Proof: As in [25], Theorem 2.7. ■

A. Stability despite switching delay

To determine whether (12) will be violated if a mode switch occurs at time τ, we bound the possible variations in Lyapunov functions during the switching delay, and define a set of switching signals Σ^{T_D} such that (6) is GUAS under Σ^{T_D} despite the switching delay T_D.

Lemma 1: For a switched system (4), assume that $\sigma(t) \equiv p$ for $t \in [\tau - T_D, \tau)$, $\tau \geq T_D$. Then there exists $c_i, \mu_i > 0$ such that for all $t \in [\tau - T_D, \tau)$,

$$\|x_i(t)\| \leq c_i e^{-\mu_i(t-(\tau-T_d))} \|x_i(\tau - T_D)\| \tag{13}$$

for any x_i corresponding to the i^{th} block diagonal subsystem.

Proof: Consider $\|[x_{i+1}^T, ..., x_N^T]^T\|$ an exponentially decaying input to $\dot{x}_i$, and apply Remark 1. ■

Theorem 3: Let Σ^{T_D} be the set of piecewise constant switching signals $\sigma(\cdot) : [0, \infty) \to \mathcal{P}$ such that, for each switching instant τ, $x(\tau - T_D) \in \mathcal{S}(\sigma(\tau^-), \sigma(\tau), \tau)$, where

$$\mathcal{S}(p, q, \tau) \quad := \quad \left\{ x \in \mathbb{R}^n : \frac{x^T(P_p - P_q)x}{\|x\|^2} > \gamma(p, q, \tau) \right\}, \tag{14}$$

$$\gamma(p, q, \tau) \quad = \quad \Theta(p, \tau) \cdot \Lambda(p, q) \tag{15}$$

with $\Theta(p, \tau) = \frac{c_p^2 e^{-2\lambda_p \tau}}{2\lambda_p}(e^{2\lambda_p T_D} - 1)$, $\Lambda(p, q) = \lambda_{\max}(Q_p(p)) - \min(0, \lambda_{\min}(Q_q(p)))$, and $c_p, \lambda_p > 0$ are constants determined by Definition 2 for mode p. Then (6) is GUAS under Σ^{T_D}.

Proof: By Theorem 2, a sufficient condition for $\sigma(\cdot) \in \Sigma^*$ is that for all τ with $\sigma(\tau^-) = p$ and $\sigma(\tau) = q$,

$$V_p(x(\tau)) - V_q(x(\tau)) > 0 \tag{16}$$

We show that $\Sigma^{T_D} \subseteq \Sigma^*$ by finding a lower bound for (16) given only $x(\tau - T_D)$, and partitioning the state space accordingly. Since

$$\begin{aligned} V_p(x(\tau)) = &\quad V_p(x(\tau - T_D)) + \int_{\tau-T_D}^{\tau} \dot{V}_p(x(t))dt \\ = &\quad V_p(x(\tau - T_D)) - \int_{\tau-T_D}^{\tau} x^T(t) Q_p(p) x(t) dt \end{aligned} \tag{17}$$

and $\lambda_{\min}(Q_p(p)) > 0$, and by the Courant-Fischer theorem $\lambda_{\min}(Q_p(p))\|x\|^2 \leq x^T Q_p(p) x \leq \lambda_{\max}(Q_p(p))\|x\|^2$, we obtain obtain a lower bound

$$\begin{aligned} V_p(x(\tau)) \quad \geq &\quad V_p(x(\tau - T_D)) \\ &\quad - \int_{\tau-T_D}^{\tau} \lambda_{\max}(Q_p(p)) \|x(t)\|^2 dt \\ \geq &\quad V_p(x(\tau - T_D)) \\ &\quad - \Theta(p, \tau) \lambda_{\max}(Q_p(p)) \|x(\tau - T_D)\|^2 \end{aligned} \tag{18}$$

using Lemma 1. Similarly, to find an upper bound,

$$\begin{aligned} V_q(x(\tau)) = &\quad V_q(x(\tau - T_D)) - \int_{\tau-T_D}^{\tau} x^T(t) Q_p(q) x(t) dt \\ \leq &\quad V_q(x(\tau - T_D)) \\ &\quad - \int_{\tau-T_D}^{\tau} \lambda_{\min}(Q_p(q)) \|x(t)\|^2 dt \end{aligned} \tag{19}$$

If $\lambda_{\min}(Q_p(q)) < 0$, the integral term is positive, and we use the upper bound for $\|x(t)\|$ given by Lemma 1 to obtain a result similar to (18). However, if $\lambda_{\min}(Q_p(q)) \geq 0$, the integral term is negative, and we require a lower bound for $\|x(t)\|$ to further bound (18). In general, such a lower bound is unavailable, but can be conservatively approximated as 0. Hence an upper bound for (19) is

$$V_q(x(\tau)) \leq V_q(x(\tau - T_D)) - \Theta(p, \tau) \Gamma(p, q) \|x(\tau - T_D)\|^2 \tag{20}$$

with $\Gamma(p, q) = \min(0, \lambda_{\min}(Q_p(q)))$. Combining (18), (20), and (16),

$$V_p(x(\tau - T_D)) - V_q(x(\tau - T_D)) > \gamma(q, p, \tau) \cdot \|x(\tau - T_D)\|^2 \tag{21}$$

Defining $\mathcal{S}(p, q, \tau)$ as the subset of $\mathbb{R}^n$ where (21) holds, we obtain (14). For any piecewise constant switching signal $\sigma(\cdot) \in \Sigma^{T_D}$, we have $\sigma(\cdot) \in \Sigma^*$, thus $\Sigma^{T_D} \subseteq \Sigma^*$. ■

The sets $\mathcal{S}(p, q, \tau)$ thus partition the state-space into regions where switching from mode p to mode q will not violate (12) despite a switching delay T_D.

Remark 3: For $\lambda_{\min}(P_p - P_q) > \gamma(p, q, \tau)$, $\mathcal{S}(p, q, \tau) = \mathbb{R}^n$. Similarly, for $\lambda_{\max}(P_p - P_q) < \gamma(p, q, \tau)$, $\mathcal{S}(p, q, \tau) = \emptyset$. By computing the minimum (maximum) eigenvalues beforehand, it can quickly be determined if a given mode switch at time $t = \tau$ will always (never) be guaranteed to maintain GUAS despite a switching delay.

If $T_D = 0$, then $\gamma(p, q, \tau) = 0$ and $\mathcal{S}(p, q, \tau)$ corresponds to a partitioning of the state-space according to (12), that we denote $\bar{\mathcal{S}}(q, p)$. From (15), if no switch occurs, i.e. $\tau \to \infty$, then $\gamma(p, q, \tau) \to 0$, or equivalently $\mathcal{S}(p, q, \tau) \to \bar{\mathcal{S}}(p, q)$.

B. Wait-time condition

We now fix a "next mode" q and study the evolution of $\mathcal{S}(\sigma(t), q, t)$ under a switching signal $\sigma(\cdot) \in \Sigma^*$ to determine how these regions evolve over time. We define the functional $\gamma(\cdot, q, \cdot) : \mathcal{P} \times [0, \infty) \to \mathbb{R}$, evolving under a switching signal $\sigma(\cdot) \in \Sigma^*$, as

$$\gamma(\sigma(t), q, t) \ = \Theta(\sigma(t), t) \cdot \Lambda(\sigma(t), q) \tag{22}$$

Corollary 2: For $\sigma(\cdot) \in \Sigma^{T_D}$, as $t \to \infty$, $\mathcal{S}(p, q, \tau) \to \bar{\mathcal{S}}(p, q)$ for all $p, q \in \mathcal{P}$.

Proof: For all $p, q \in \mathcal{P}$, $t \geq T_D$, $\gamma(p, q, t) \geq 0$. It follows that $\gamma(\sigma(t), q, t) \geq 0$ for all $\sigma(\cdot) \in \Sigma^{T_D}$. Hence

$$0 \quad \leq \quad \gamma(\sigma(t), q, t) \leq \Theta_{\max} \cdot \Lambda_{\max} \tag{23}$$

with $\Theta_{\max} = \max_{\sigma(t)} \Theta(\sigma(t), T_D)$, and $\Lambda_{\max} = \max_{\sigma(t)} \Lambda(\sigma(t), q)$. The exponential decay term in $\Theta_{\max}$ ensures that $\gamma(\sigma(t), q, t) \to 0$ as $t \to \infty$ for all $\sigma(\cdot) \in \Sigma^{T_D}$. Wtih p the final active mode of $\sigma(\cdot)$, the result follows. ■

According to Corollary 2, by waiting long enough before switching, the effect of the time delay on the state-based partitioning can be made arbitrarily small. Specifically, if for some $\tau^*, \gamma^* \in \mathbb{R}$, $\gamma(p, q, \tau^*) < \gamma^*$, then $\gamma(p, q, t) < \gamma^*$ for all $t \geq \tau^*$. Once this condition is satisfied, it is satisfied for all future times, and hence can be thought of as a *wait-time* condition. In contrast, the dwell-time condition presented in [24] must be satisfied after each mode switch in order to guarantee asymptotic stability of a switched linear system. Wait-time instead provides a time τ^* after which the effect of the time delay T_D on the partitioning (12) can be ignored.

The wait-time condition has a very important consequence when applied to UAV systems: often times a UAV will operate in a given mode for an extended period of time (e.g. "return to base") before switching through other modes more rapidly (e.g. "lower landing gear", "change flaps configuration", "acquire glide slope"). However, since wait-time is not reset after each mode change, the effect of $\gamma(p, q, t)$ on the state partitions during the final sequence of rapid mode switches will already have decayed to a negligible level.

V. Examples

A. Cooperative control of vehicle formations

Recall the fleet of vehicles (3) with block diagonal subsystems (7) under arbitrary switching. Applying Theorem 1, LMI solvers can be employed on the block diagonal subsystems, rather than to the entire system, to show GUAS under arbitrary switching. For a P mode system, we have thus reduced the problem to N sets of $P+1$ LMIs in $\mathbb{R}^{2\times 2}$ as opposed to $P+1$ LMIs in $\mathbb{R}^{2N\times 2N}$.

1) Five vehicle fleet: Consider a five vehicle system with

$$A_{\text{cl}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \begin{aligned} K_1 &= [-20 \ \ -5] \\ K_2 &= [-4 \ \ -6] \end{aligned} \tag{24}$$

and

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix} \tag{25}$$

switching randomly between K_1 and K_2. The block diagonal subsystems are given by (7), with $\lambda_i \in \{3.0108, 4.6180, 4.6180, 2.3819, 2.3819\}$. For each of the five block diagonal subsystems, a GQLF was found by solving three LMIs in $\mathbb{R}^{2\times 2}$ to obtain a symmetric positive definite matrix P_i.

$$\begin{aligned} P_1 &= \begin{bmatrix} 88.8184 & -27.8822 \\ -27.8822 & 64.5339 \end{bmatrix} \\ P_2 = P_3 &= \begin{bmatrix} 0.0041 & -0.0099 \\ -0.0099 & 0.0831 \end{bmatrix} \\ P_4 = P_5 &= \begin{bmatrix} 1.6822 & -3.3996 \\ -3.3996 & 22.0348 \end{bmatrix} \end{aligned} \tag{26}$$

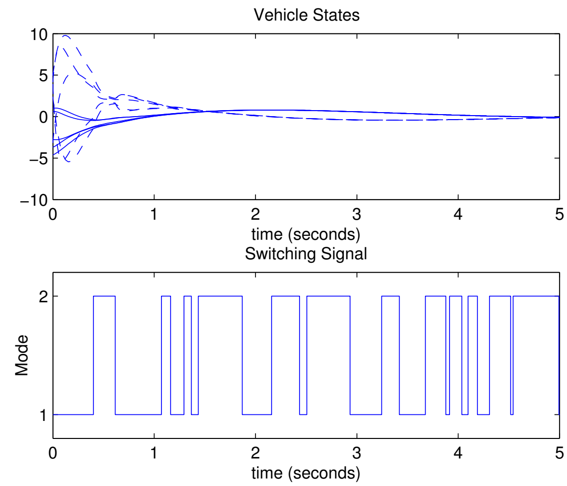

Fig. 2. Simulation results for the five vehicle system given by (3), (24) and Laplacian (25), with mode switches occurring according to the arbitrary switching signal σ shown in the bottom plot. By Corollary 1, GUAS under arbitrary switching for (3) was proven by showing GUAS under arbitrary switching for each block diagonal subsystem (7). Shown are the position (top plot, solid) and velocity (top plot, dashed) variables of each of the vehicles.

By Corollary 1, the GQLFs $V^i(x_i) = x_i^T P_i x_i$, $i \in \{1, \cdots, 5\}$ prove GUAS under arbitrary switching for the full 10-dimensional system. Figure 2 shows simulation results (top plot) for an arbitrary switching signal (bottom plot).

2) 100 vehicle fleet: Consider a 100 vehicle system with the same A_{cl}, K_1 and K_2 as in (24). The Laplacian L (not presented) is normalized such that all of its eigenvalues lie within a disk of radius 1 centered at $1 + 0j$ in the complex plane (cf. Proposition 2 in [5]), and strongly connected such that its zero eigenvalue is simple (cf. Proposition 3 of [5]), a necessary condition for the stability of such systems. For both K_1 and K_2, the block diagonal subsystems given by (7) are stable for all eigenvalues of L.

Solving three LMIs in $\mathbb{R}^{200\times 200}$ in Matlab was not possible on a dual core 2.40Ghz Intel-based machine with 4GB RAM due to the dimensionality of the system and ensuing memory requirements. Exploiting Corollary 1, we solve 100 sets of three LMIs to obtain a GQLF for each block diagonal subsystem and prove GUAS under arbitrary switching. The LMIs solved are in $\mathbb{R}^{2\times 2}$ for block diagonal subsystems for which λ_i is real, and are in $\mathbb{R}^{4\times 4}$ when λ_i is complex.

B. Remote Supervisory Control of a Switched Linear System

Consider a two mode switched linear system

$$\dot{x} = A_\sigma x \tag{27}$$

with switching delay $T_D = 0.1$ s, and $\mathcal{A} := \{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} \tag{28}$$

Applying the Converse Lyapunov theorem, it can be shown that no GQLF exists for this switched linear system. According to Theorem 3, we partition the state space into regions

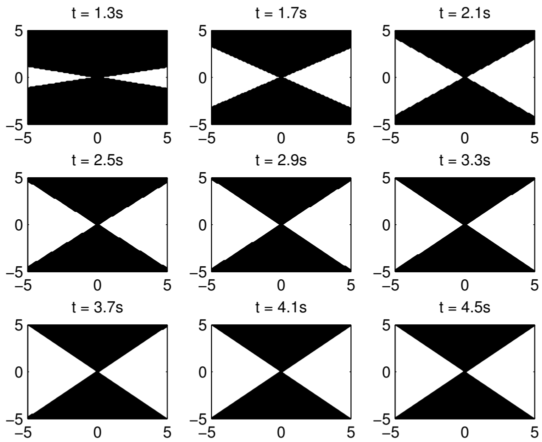

Fig. 3. Snapshots of $\mathcal{S}(1,2,t)$ (white) evolving over time under the switching signal $\sigma(t) \equiv 1$. $\mathcal{S}(1,2,t)$ converges to the standard Lyapunov based partitioning $\bar{\mathcal{S}}(1,2)$.

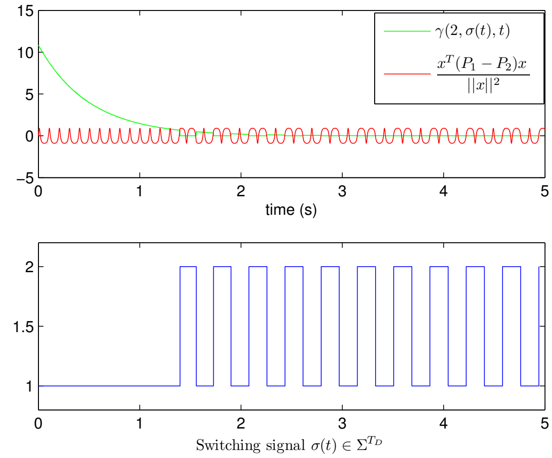

Fig. 4. Evolution of $\gamma(\sigma(t),2,t)$ and the switching signal generated by switching whenever possible without violating the constraints imposed by Theorem 3. Initially $\gamma(1,2,t) > \lambda_{\max}(P_1 - P_2)$ is too large to allow any mode switches, and by Remark 3, $\mathcal{S}(1,2,t) = \emptyset$. After approximately 1.3s, $\gamma(1,2,t) < \lambda_{\max}(P_1 - P_2)$, and a mode switch is triggered as soon as the delayed trajectory $x(t-T_D)$ enters $\mathcal{S}(1,2,t)$.

$\mathcal{S}(1,2,t)$ and $\mathcal{S}(2,1,t)$, which provide switching restrictions to maintain GUAS despite a switching delay. Figure 3 shows snapshots of $\mathcal{S}(1,2,t)$ (white) evolving over time under the switching signal $\sigma(t) \equiv 1$. $\mathcal{S}(1,2,t)$ converges to the standard Lyapunov based partitioning $\bar{\mathcal{S}}(1,2)$.

Figure 4 shows the evolution of $\gamma(\sigma(t),2,t)$ overlaid with $V_{12}(t) := \frac{x(t)^T(P_1-P_2)x(t)}{\|x(t)\|^2}$, and the switching signal $\sigma(\cdot) \in \Sigma^{T_D}$ generated by switching whenever possible without violating the constraints imposed by Theorem 3. Initially $\gamma(1,2,t) > \lambda_{\max}(P_1 - P_2)$ is too large to allow any mode switches, and by Remark 3, $\mathcal{S}(1,2,t) = \emptyset$. After approximately 1.3s, $\gamma(2,1,t) < \lambda_{\max}(P_1 - P_2)$, and a mode switch is triggered as soon as the delayed trajectory $x(t-T_D)$ enters $\mathcal{S}(1,2,t)$. In Figure 5, we zoom in on when $x(t-T_D) \in \mathcal{S}(1,2,t)$ for the first time, at $t = \tau$. Clearly, $V_{12}(\tau) > 0$, satisfying the stability requirements imposed by Theorem 2.

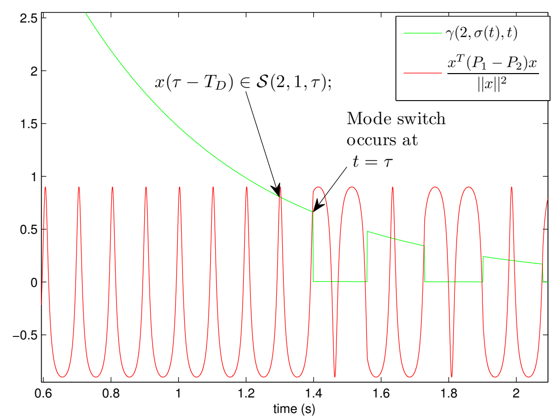

Fig. 5. From Figure 4, a close view of when $x(\tau - T_D) \in \mathcal{S}(1,2,\tau)$ for the first time at $t = \tau$. Clearly, $V_{12}(\tau) > 0$, satisfying the stability requirements imposed by Theorem 2.

VI. Conclusion and Future Work

Motivated by problems in distributed systems such as fleets of remotely supervised vehicles, we consider switched linear systems with switching delay. Communication delays and human response time could potentially destabilize the system, hence this paper presents an attempt to compensate a priori, in the design stage, for potential delays in large distributed systems.

We exploited the structure of block upper-triangular switched linear systems to show that a system is GUAS under a given switching scheme if and only if each of its block diagonal subsystems is GUAS under this switching scheme as well. As a consequence, only the subset of the state space corresponding to the block diagonal subsystems that are not GUAS under arbitrary switching need to be considered when synthesizing state constraint based switching signals. A scalable, computationally efficient LMI-based test for GUAS under arbitrary switching enables proofs of stability for large systems (e.g., 100 vehicles). In the case in which a common quadratic Lypaunov function does not exist for one or more subsystems, we presented a method of determining state constraints to ensure GUAS despite a switching delay. These constraints asymptotically converge to the standard (delay-free) Lyapunov-based state constraints for stability.

The case of addressing fleets of either non-identical or nonlinear systems is much more difficult, and one potential avenue of future work. The properties of the Kronecker product break down, and the system can no longer be transformed into block upper-triangular form, although some preliminary results based on Lyapunov and optimal control theory do exist. We also hope to consider the restriction of simultaneous mode switches among subsystems, as in real-world applications this may prove to be an unrealistic assumption.

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