# Overapproximating the Reachable Sets of LTI Systems Through a Similarity Transformation

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*Abstract*— **We present a decomposition method for complexity reduction in reachability analysis and controller synthesis based on a series of transformations. The decomposition is guaranteed to yield weakly-coupled (lower dimensional) subsystems with disjoint control input across them. Reachable sets, computed independently for each subsystem, are back-projected and intersected to yield an overapproximation of the actual reachable set. Using an example we show that significant reduction in the computational costs can be achieved. This technique has considerable potential utility for use in conjunction with computationally intensive reachability tools.**

*Keywords:* **reachability analysis, dimension reduction, transformation, projection, LTI systems, decomposition**

## I. INTRODUCTION

A major step towards formal verification and controller synthesis of continuous and hybrid systems is the so-called reachability analysis. Historically, a major obstacle in employing reachability analysis has been the "curse of dimensionality" [1]. The computational complexity of reachability techniques increases with the dimension of the continuous state space, often rendering them impractical for complex real-life applications. This difficulty motivated the development of more efficient reachability techniques within the past few years [2], [3]. Despite their success, the applicability of these methods is limited to systems whose constraints can be described by very specific classes of shapes (e.g., ellipsoids and zonotopes) in both the input and the state spaces. On the other hand, for many applications, the ability to take advantage of some of the unique features (e.g., safety controller synthesis, and handling of non-convex or arbitrarily shaped sets) offered almost exclusively by more computationally intensive reachability tools is of critical importance.

This paper focuses on continuous linear time-invariant (LTI) systems (and by extension, hybrid systems with LTI continuous dynamics). We aim to broaden the range of applicable reachability tools for LTI systems with high dimensionality, to enable the use of reachability tools that would otherwise be too computationally complex to employ (e.g., [4], [5], [6]). We accomplish this through a series of transformations of the system into a coordinate space in which reachability could be performed in lower-dimensional subspaces and is guaranteed to yield an overapproximation of the actual reachable set in that space. Performing reachability in lower dimensions, we obtain significant reduction in the computational costs regardless of the reachability tool used.

Complexity reduction for reachability analysis has been addressed by a number of researchers. In general, methods to compute reachable sets for higher dimensional systems can be divided into three categories. First are techniques that take advantage of certain representations of sets in the state space [2], [3], [7]. Second are techniques that make use of model reduction and approximation [8], [9], hybridization [10], projection [11] and structure decomposition [12], [13], [14]. Finally, third are methods that combine the approaches from the first two categories. For instance, [15] employs both model approximation (through Krylov subspace projection) and efficient set representation (using low-dimensional polytopes) to perform reachability for very large-scale systems with affine dynamics.

In [11], a projection scheme based on Hamilton-Jacobi-Isaacs (HJI) PDEs is considered in which the projection of the actual reachable set is overapproximated in lower dimensional subspaces where the unmodeled dimensions are treated as disturbance. Similarly in concept, [12] decomposes a full-order nonlinear system to either disjoint or overlapping subsystems and solves multiple HJI PDEs in lower dimensions. The computed reachable set for each subsystem is an overapproximation of the projection of the full-order reachable set onto the subsystem's subspace. In [13], using an $\epsilon$-decomposition procedure, affine systems are decomposed into multiple subsystems and reachability is performed on each lower-dimensional subsystem. In [14] we presented a Schur-based decomposition for LTI systems. A Sylvester equation (or an optimization problem) was solved in order to eliminate (minimize) the coupling between the resultant subsystems. Additional constraints were imposed when the control input was non-disjoint across candidate subsystems. Reachability was then performed on each subsystem independently.

In this paper, our main contribution is to provide an additional method, based on structure decomposition, to reduce the complexity of reachability analysis for high dimensional LTI systems. We present a set of transformations through which resultant subsystems are ensured to a) be weakly coupled and b) have disjoint inputs. In Section II, we provide necessary preliminaries. Section III presents the decomposition method. An extension to hybrid systems is also provided. Section IV demonstrates our method on a

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numerical example. Lastly, we provide concluding remarks in Section V.

## II. Mathematical Preliminaries

We focus on LTI systems of the form

$$\dot{x} = Ax + Bu \tag{1}$$

described in standard notation by

$$G := \left[\begin{array}{c|c} A & B \\ \hline * & * \end{array}\right] \tag{2}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, state vector $x(t) \in \mathbb{R}^n$, and control input $u(t) \in \mathcal{U} \subset \mathbb{R}^p$ (with $\mathcal{U}$ a compact set). Here, "$*$" is simply a place holder for the terms in (2) with which we are not concerned in this paper.

Consider the following two definitions of reachable sets.

*Definition 1:* Given a target (unsafe) set of states $\mathcal{X}_f \subset \mathbb{R}^n$ and the time interval $\tau \in [t, t_f]$, the backward reachable set of system (2) at time $t$ is defined as $\mathcal{X}_t := Reach(\mathcal{X}_f)$, $\mathcal{X}_t \subseteq \mathbb{R}^n$ and is the set of all states for which there exists a trajectory $x(\tau)$ such that $x(t_f) \in \mathcal{X}_f$ for all control input $u(\tau) \in \mathcal{U}$.

*Definition 2:* Given a target (unsafe) set of states $\mathcal{X}_f \subset \mathbb{R}^n$ and the time interval $\tau \in [t, t_f]$, the backward reachable set of the perturbed system $\dot{x} = Ax + Ld$, $L \in \mathbb{R}^{n\times q}$, $d \in \mathcal{D} \subset \mathbb{R}^q$, at time $t$ is defined as $\mathcal{X}_t := Reach(\mathcal{X}_f)$, $\mathcal{X}_t \subseteq \mathbb{R}^n$ and is the set of all states for which there exists a trajectory $x(\tau)$ and a disturbance signal $d(\tau) \in \mathcal{D}$ such that $x(t_f) \in \mathcal{X}_f$.

For brevity, we adapt the following notation: A non-subscripted norm $\|\cdot\|$ denotes an infinity norm. In particular, for a matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ this norm is an induced norm defined by $\|A\| := \sup_{\mathrm{v}\neq 0} \frac{\|A\mathrm{v}\|}{\|\mathrm{v}\|}$, $\mathrm{v} \in \mathbb{R}^n$, and can be computed as $\max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$. For a Lebesgue measurable function $x : \mathbb{R} \to \mathbb{R}^n$ defined over an interval $[t_0, t_f]$, we denote $\|x(t)\| := \|x(t)\|_{\mathcal{L}_\infty[t_0,t_f]} = \sup_{t\in[t_0,t_f]} |x(t)| < \infty$.

Now consider the following definitions.

*Definition 3:* The LTI system that consists of two subsystems

$$\dot{x}_1 = A_1 x_1 + \Lambda_c x_2 \tag{3}$$
$$\dot{x}_2 = A_2 x_2 \tag{4}$$

with $A_1 \in \mathbb{R}^{k\times k}$, $A_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, $\Lambda_c \in \mathbb{R}^{k\times(n-k)}$, $x_1(t) \in \mathbb{R}^k$, and $x_2(t) \in \mathbb{R}^{(n-k)}$, is said to be *unidirectionally coupled* since the trajectories of (3) are affected by those of (4), while (4) evolves independently from (3).

*Definition 4:* Let there be a non-singular transformation matrix $T \in \mathbb{R}^{n\times n}$, such that $[z_1^T, z_2^T]^T = T^{-1}[x_1^T, x_2^T]^T$, and

$$\dot{z}_1 = A_1 z_1 + \tilde{\Lambda}_c z_2 \tag{5}$$
$$\dot{z}_2 = A_2 z_2. \tag{6}$$

Then (5) and (6) are said to be *unidirectionally weakly-coupled* (in comparison to (3) and (4)) if

$$\|\tilde{\Lambda}_c\| < \|\Lambda_c\|. \tag{7}$$

*Definition 5:* Let there be a non-singular transformation matrix $T \in \mathbb{R}^{n\times n}$ and a coordinate space $w := T^{-1}x$ in which (1) can be partitioned into $N$ subsystems as

$$\dot{w}_i = \tilde{A}_i w_i + \tilde{B}_i u_i, \quad i = 1, \ldots, N. \tag{8}$$

The input $u$ is *disjoint* across these subsystems if

$$u_i \in \mathcal{U}_i \subset \mathbb{R}^{p_i}, \quad p = \sum_{i=1}^{N} p_i \tag{9}$$

so that the partitioning of $\mathcal{U}$ is mutually exclusive and exhaustive.

*Definition 6:* A subsystem $i$ in (8) is said to be *trivially-uncontrollable* if it possesses a null input matrix, i.e. $\tilde{B}_i = \mathbf{0}$.

Next, consider the following lemmas which will be used in Section III.

*Lemma 1:* The Sylvester equation

$$EX + XF + H = \mathbf{0}, \tag{10}$$

with $E \in \mathbb{R}^{k\times k}$, $F \in \mathbb{R}^{m\times m}$, and $H \in \mathbb{R}^{k\times m}$, has a unique solution $X \in \mathbb{R}^{k\times m}$ if and only if the eigenvalue sum $\lambda_i(E) + \lambda_j(F) \neq 0$, $\forall i \in \{1, ..., k\}$ and $\forall j \in \{1, ..., m\}$.

*Proof:* cf. [16, Lem. 2.7]. ■

*Lemma 2 (Schur form):* For any real matrix $M \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $U^T M U = \tilde{M}$ is upper (quasi) triangular, and the eigenvalues of $M$ are the eigenvalues of the block diagonals (each of dimension 2 or less) of $\tilde{M}$. Furthermore, the matrix $U$ can be chosen to order the eigenvalues arbitrarily.

*Proof:* cf. [17, Thm's 7.1.3 and 7.4.1] and [18, 5R]. ■

*Remark 1:* There always exists a partitioning of $\tilde{M}$ such that $\tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \mathbf{0} & \tilde{M}_{22} \end{bmatrix}$.

The condition number of a non-singular matrix $\mathfrak{A} \in \mathbb{R}^{n\times n}$ is $\kappa(\mathfrak{A}) := \|\mathfrak{A}\|_2 \|\mathfrak{A}^{-1}\|_2 = \frac{\sigma_{\max}(\mathfrak{A})}{\sigma_{\min}(\mathfrak{A})}$ where $\|\cdot\|_2$ denotes the Euclidean norm and $\sigma(\cdot)$ is the singular value operator.

*Lemma 3:* Let $\mathfrak{A}_\imath, \mathfrak{A}_\jmath$ be non-singular matrices in $\mathbb{R}^{n\times n}$. Then the following properties hold:

i) $\kappa(\mathfrak{A}_\imath \mathfrak{A}_\jmath) \leq \kappa(\mathfrak{A}_\imath)\kappa(\mathfrak{A}_\jmath)$
ii) $\kappa(\mathfrak{A}_\imath \mathfrak{A}_\jmath) = \kappa(\mathfrak{A}_\imath)$ if $\mathfrak{A}_\jmath$ is orthogonal ($\mathfrak{A}_\jmath^T = \mathfrak{A}_\jmath^{-1}$).

*Proof:* The proof follows directly from the definition of $\kappa(\cdot)$ above and noting that $\|\mathfrak{A}_\imath \mathfrak{A}_\jmath\|_2 \leq \|\mathfrak{A}_\imath\|_2 \|\mathfrak{A}_\jmath\|_2$ and that if $\mathfrak{A}_\jmath^T = \mathfrak{A}_\jmath^{-1}$, then $\|\mathfrak{A}_\jmath\|_2 = 1$. ■

*Remark 2 ([19]):* If $\mathfrak{A} \in \mathbb{R}^{n\times n}$ is a transformation matrix, $\kappa(\mathfrak{A})$ measures the degree of distortion of $\mathfrak{A}x$, $x \in \mathbb{R}^n$.

Finally, a linear transformation of a set $\mathcal{X} \subseteq \mathbb{R}^n$ using an invertible transformation matrix $T \in \mathbb{R}^{n\times n}$ is $\mathcal{V} := \{v \in \mathbb{R}^n \mid v = T^{-1}x,\ x \in \mathcal{X}\}$. This, with an abuse of notation, is sometimes stated as $\mathcal{V} = T^{-1}\mathcal{X}$.

## III. Problem Formulation and Solution Methodology

In this section we present a decomposition method which results in unidirectionally weakly coupled subsystems with disjoint input across them. Reachability analysis can then be performed on these lower dimensional subsystems instead of the full-order system.

We assume a partitioning of (2) that results in exactly two subsystems. However, the proposed method is generalizable to $N$ subsystems by applying the same decomposition technique to each subsystem iteratively. A higher number of subsystems (i.e. iterated decomposition) may result in a more conservative overapproximation of the actual reachable set.

When the control input is non-disjoint, even if the dynamics of the subsystems are completely decoupled, their evolution is tightly paired through a common input. The difficulty arises, for example, when in reachability computation a control value deemed optimal for one subsystem is in fact non-optimal for the full-order system. Blindly performing reachability for each subsystem separately may result in an underapproximation and additional measures have to be taken in order to ensure the overapproximation of the actual reachable set.

In [14] we proposed a modified transformation that ensured a decomposition with disjoint input across the computed subsystems. However, one such transformation could, in some cases, increase the unidirectional coupling between subsystems which in turn would potentially result in excessive overapproximation of the full-order reachable set.

*Here we present a series of transformations that simultaneously address the above issues. That is, the resultant subsystems are unidirectionally* weakly *coupled across which the input is guaranteed to be disjoint by ensuring that one of the subsystems in the new coordinate system is trivially-uncontrollable.*

In such a case, as discussed in [14], it is clear that the (otherwise non-disjoint) control action does not affect the evolution of the reachable set of the trivially-uncontrollable subsystem. Therefore, an optimal control input for the subsystem with nonzero input matrix is also optimal for the full-order system.

### A. The Similarity Transformation

We first apply Lemma 2 with an orthogonal transformation matrix $T_1 \in \mathbb{R}^{n\times n}$ to (2) to obtain

$$G' = T_1^{-1}(G) = \left[\begin{array}{c|c} T_1^T A T_1 & T_1^T B \\ \hline * & * \end{array}\right] = \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \multicolumn{2}{c|}{*} & * \end{array}\right] \quad (11)$$

with $\tilde{A}_{11} \in \mathbb{R}^{k\times k}$, $\tilde{A}_{12} \in \mathbb{R}^{k\times(n-k)}$, $\tilde{A}_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, $\tilde{B}_1 \in \mathbb{R}^{k\times p}$, and $\tilde{B}_2 \in \mathbb{R}^{(n-k)\times p}$.

Applying a second transformation

$$T_2 = \begin{bmatrix} \alpha I_{k\times k} & \mathbf{0} \\ \mathbf{0} & I_{(n-k)\times(n-k)} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad (12)$$

with $\alpha \in \mathbb{R}$, $\alpha > 1$ results in

$$G'' = T_2^{-1}(G') = \left[\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \multicolumn{2}{c|}{*} & * \end{array}\right]. \quad (13)$$

In this case if $\alpha$ is chosen to be sufficiently large, the subsystems in (13), collectively written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & A_c \\ \mathbf{0} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \alpha^{-1}\tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \quad (14)$$

with $z = (T_1 T_2)^{-1} x$ and $A_c := \alpha^{-1}\tilde{A}_{12}$, can be made unidirectionally weakly coupled (since $\|A_c\| < \|\tilde{A}_{12}\|$) with "nearly" disjoint input decomposition across them. However, such a transformation with a deliberately large $\alpha \gg 1$, among other shortcomings, is prone to cause numerical instability in reachability computations since the target set in the transformed coordinates, $\mathcal{Z}_f = (T_1 T_2)^{-1}\mathcal{X}_f$, is likely to become too severely distorted to be of any practical use.

We address this issue by using a third transformation, subject to the assumption that $\tilde{B}_2$ is full-column rank. Notice that this assumption is not too restrictive and is generally satisfied for $(n-k) \geq p$.

*Theorem 1:* Let $\tilde{B}_2^\dagger := (\tilde{B}_2^T \tilde{B}_2)^{-1}\tilde{B}_2^T$ be the left inverse of $\tilde{B}_2$. Then a nonsingular transformation matrix

$$T_3 = \begin{bmatrix} I_{k\times k} & \alpha^{-1}\tilde{B}_1\tilde{B}_2^\dagger \\ \mathbf{0} & I_{(n-k)\times(n-k)} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad (15)$$

with a bounded $\alpha$ given as

$$\alpha = \max\left(1\,,\, \frac{\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\|}{\|\tilde{A}_{12}\|}\right) + \epsilon \quad (16)$$

and a small $\epsilon \in \mathbb{R}^+$, makes the subsystems in (14) unidirectionally weakly coupled across which the input is completely disjoint.

*Proof:* Applying the transformation $T_3$ to (13) we obtain

$$G''' = T_3^{-1}(G'') = \left[\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \multicolumn{2}{c|}{*} & * \end{array}\right] \quad (17)$$

with

$$\Pi = \alpha^{-1}\big(\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\big). \quad (18)$$

Thus, the first subsystem has been made trivially-uncontrollable regardless of the value of $\alpha$. Disjoint decomposition of the control input across subsystems follows naturally.

To prove that the value of $\alpha$ given in (16) will result in subsystems that are unidirectionally weakly coupled while preventing the severe distortion of the target set (as discussed previously), we solve the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \alpha \\ \text{subject to} \quad & \|\Pi\| < \|\tilde{A}_{12}\| \\ & \alpha > 1 \end{aligned} \quad (19)$$

That is, we seek a value for $\alpha$ that is as small as possible to prevent potential numerical difficulties but is large enough to ensure weakening of the unidirectional coupling.

Luckily, this problem has a closed-form solution. Consider the first constraint. Simple algebraic manipulation gives,

$$|\tfrac{1}{\alpha}| \cdot \|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\| < \|\tilde{A}_{12}\|$$

$$\Rightarrow \alpha > \frac{\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\|}{\|\tilde{A}_{12}\|} \tag{20}$$

Therefore, any $\alpha$ that satisfies both (20) and $\alpha > 1$, that is,

$$\alpha > \max\left(1\,,\frac{\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\|}{\|\tilde{A}_{12}\|}\right) \tag{21}$$

is a feasible solution, i.e. a solution that weakens the coupling between the two subsystems. However, we resort to

$$\alpha = \max\left(1\,,\frac{\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\|}{\|\tilde{A}_{12}\|}\right) + \epsilon \tag{22}$$

with a small $\epsilon \in \mathbb{R}^+$ in order to solve (19) and at the same time avoid potential numerical issues that may be caused by large values of $\alpha$. ■

But how large can $\alpha$ (as given in Theorem 1) be? Consider

$$\mu := \frac{\|\tilde{A}_{11}(\tilde{B}_1\tilde{B}_2^\dagger) - (\tilde{B}_1\tilde{B}_2^\dagger)\tilde{A}_{22} + \tilde{A}_{12}\|}{\|\tilde{A}_{12}\|} \tag{23}$$

in more detail. We formalize an upper bound on $\mu$ in terms of the distance of $\tilde{B}_1\tilde{B}_2^\dagger$ from the optimal solution

$$X = \underset{Q\in\mathbb{R}^{k\times(n-k)}}{\arg\min}\ \|\tilde{A}_{11}Q - Q\tilde{A}_{22} + \tilde{A}_{12}\|. \tag{24}$$

Notice that under the conditions in Lemma 1, $X$ is simply the solution of the Sylvester equation $\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} = \mathbf{0}$.

*Lemma 4:* Let $X$ be given as in (24). Then,

$$\|\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12}\| \le \|\tilde{A}_{12}\|. \tag{25}$$

*Proof:* By contradiction. The hypothesis $\|\tilde{A}_{12}\| < \|\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12}\|$ would imply that $X = \mathbf{0}$ can never be a solution. Since there are no constraints in (24) imposing this restriction, we conclude that $\|\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12}\| \le \|\tilde{A}_{12}\|$. ■

Now define an auxiliary variable

$$\Delta := \tilde{B}_1\tilde{B}_2^\dagger - X. \tag{26}$$

Substituting for $\tilde{B}_1\tilde{B}_2^\dagger$ in (23) and using results of Lemma 4 we obtain

$$\begin{aligned}\mu &\le \frac{\|\tilde{A}_{11}\Delta - \Delta\tilde{A}_{22}\| + \|\tilde{A}_{12}\|}{\|\tilde{A}_{12}\|}\\ &\le \|\Delta\| \cdot \left(\frac{\|\tilde{A}_{11}\| + \|\tilde{A}_{22}\|}{\|\tilde{A}_{12}\|}\right) + 1.\end{aligned} \tag{27}$$

Consequently we can observe that

$$\lim_{\|\Delta\|\to 0} \alpha = 1 + \epsilon. \tag{28}$$

In other words, the closer $\tilde{B}_1\tilde{B}_2^\dagger$ is to the optimal solution $X$ of (24), the smaller $\alpha$ needs to be. This in turn ensures a better-conditioning of the transformation matrix

$$\mathsf{T} := T_3^{-1}T_2^{-1} = \begin{bmatrix} \alpha^{-1}I & -\alpha^{-1}\tilde{B}_1\tilde{B}_2^\dagger \\ \mathbf{0} & I \end{bmatrix} \tag{29}$$

which then causes less distortion of the transformed target set given by $T^{-1}\mathcal{X}_f$, $T = T_1T_2T_3$. (Notice that $\kappa(T^{-1}) = \kappa(\mathsf{T})$ according to Lemma 3 due to orthogonality of $T_1$.)

Finally let us remark that, if necessary, a maximal value for $\alpha$ can be prescribed by first choosing (e.g. heuristically) an appropriate/tolerable upper bound on the condition number $\kappa(\mathsf{T})$, and then increasing $\alpha$ until that bound is reached.

Notice that regardless of the value of $\alpha$, the transformations are always invertible.

### *B. Reachability in Lower Dimensions*

In the new coordinate space we now have

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \Pi \\ \mathbf{0} & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \tilde{B}_2 \end{bmatrix} u \tag{30}$$

with $y = T^{-1}x$, $T = T_1T_2T_3$, and $\Pi$ and $\alpha$ as given in (18) and (16), respectively.

It is clear that $y_2 \in \mathbb{R}^{(n-k)}$ evolves independently of $y_1 \in \mathbb{R}^k$ since

$$\dot{y}_2 = \tilde{A}_{22}y_2 + \tilde{B}_2u. \tag{31}$$

However, $y_1$ is affected by $y_2$ through $\Pi$. That is, we have

$$\dot{y}_1 = \tilde{A}_{11}y_1 + \Pi\, y_2. \tag{32}$$

Reachability analysis in $y$-coordinates can be performed on each lower-dimensional subsystem separately:

**Algorithm 1** Reachability in lower dimensions

$\mathcal{Y}_f \leftarrow T^{-1}\mathcal{X}_f$
**for** $i \leftarrow 1, 2$ **do**
  $\mathcal{Y}_f^i \leftarrow proj(\mathcal{Y}_f, i)$ ▷ project onto $i$-th subspace
**end for**
For subsystem #2:
  $\mathcal{Y}_t^2 \leftarrow Reach(\mathcal{Y}_f^2)$ ▷ using Definition 1
For subsystem #1:
  Treat $\Pi\, y_2$ as disturbance
  $\xi \leftarrow \sup_{y_2\in\mathcal{Y}_t^2}\|y_2\|$
  Compute upper-bound $\|\Pi\, y_2\| \le \|\Pi\| \cdot \xi$
  $\mathcal{Y}_t^1 \overset{\text{consrv.}}{\longleftarrow} Reach(\mathcal{Y}_f^1)$ ▷ using Definition 2
**return** $(\mathcal{Y}_t^1, \mathcal{Y}_t^2)$

Notice that in the new coordinate space, the reachable set in the subspace of the trivially-uncontrollable subsystem (32) is computed without the need for solving a differential game. In fact for this subsystem the unidirectional coupling is treated as disturbance and consequently this disturbance together with the dynamics strive to enlarge the reachable (unsafe) set as much as possible. Therefore using Cauchy's formula we have the following set-valued formulation of the reachable set in the upper subspace

$$\mathcal{Y}_t^1 = e^{\tilde{A}_{11}t}\mathcal{Y}_f^1 \oplus \int_0^t e^{\tilde{A}_{11}(t-r)}\mathcal{D}(r)dr, \quad \text{for } t \le 0 \tag{33}$$

where $\oplus$ denotes the Minkowski sum. Here the input (disturbance) draws from the set $\mathcal{D}$, a compact infinity norm ball of radius $\|\Pi\|\xi$ centered around the origin. Let $y_{1,0} \in \mathcal{Y}_f^1$ and $y_1 \in \mathcal{Y}_t^1$, then for $t \geq 0$

$$\|y_1 - e^{-\tilde{A}_{11}t}y_{1,0}\| \leq \int_0^t e^{\|\tilde{A}_{11}\|(t-r)}\|\Pi\|\xi dr \quad (34)$$

$$= \frac{e^{\|\tilde{A}_{11}\|t} - 1}{\|\tilde{A}_{11}\|}\|\Pi\|\xi \quad (35)$$

$$\leq \left(\sum_{s=1}^{\infty} \frac{t^s\big(\overline{\sigma}(\tilde{A}_{11})\sqrt{k}\big)^{s-1}}{s!}\right)\|\Pi\|\xi =: \eta \quad (36)$$

where $\overline{\sigma}(\cdot)$ is the largest singular value operator, and $k$ is the dimension of the trivially-uncontrollable subsystem. Now let $\mathcal{B}(0,\eta)$ denote an infinity norm ball of radius $\eta$ centered around the origin. Therefore we have

$$\mathcal{Y}_t^1 \subseteq e^{\tilde{A}_{11}t}\mathcal{Y}_f^1 \oplus \mathcal{B}(0,\eta), \quad \text{for } t \leq 0. \quad (37)$$

The right hand side of (37) provides an upper-bound for how much $\mathcal{Y}_t^1$ can grow in backward time. In particular, the choice of $k$, the magnitude of the unidirectional coupling $\|\Pi\|$, the supremum of the reachable set in the lower subspace $\xi$ ($=\sup_{y_2\in\mathcal{Y}_t^2}\|y_2\|$), and the largest singular value of the upper subsystem $\overline{\sigma}(\tilde{A}_{11})$ affect the conservatism of the reachable set $\mathcal{Y}_t^1$. Moreover, given $k$ and $t$, the flexibility of the Schur form in placing the eigenvalues in any order along the block-diagonals of $\tilde{A}$ can be exploited to make this subsystem evolve with slower dynamics. Through various tests we were able to confirm that doing so could potentially prevent the excessive growth of $\mathcal{Y}_t^1$ by influencing both $e^{\tilde{A}_{11}t}$ and $\eta$.

The overapproximation of the actual reachable set of the full-order system in $\mathbb{R}^n$ can be obtained using the following corollary.

*Corollary 1 (cf. [12]):* Let $\mathcal{Y}_t^i$, $i = \{1, 2\}$, denote the computed lower-dimensional overapproximative reachable set of subsystem $i$. Then the transformation of the intersection of the back-projection of these sets onto $\mathbb{R}^n$ overapproximates the actual full-order reachable set $\mathcal{X}_t$ of system (2). That is,

$$\hat{\mathcal{X}}_t := T\Big((\mathcal{Y}_t^1 \times \mathcal{R}_2) \cap (\mathcal{Y}_t^2 \times \mathcal{R}_1)\Big) \supseteq \mathcal{X}_t, \quad (38)$$

where $T = T_1T_2T_3$ is the transformation matrix and $\mathcal{R}_i$ is the appropriately dimensioned subspace of the $i$-th subsystem.

### C. Extension to Hybrid Systems

The extension of our method to hybrid dynamical systems is fairly straight forward [14]. Consider the hybrid automaton $(\mathfrak{Q}, \mathfrak{X}, f, \mathcal{U}, \Sigma, R)$, with discrete modes $\mathfrak{Q} = \{q_i\}$, continuous states $x \in \mathfrak{X}$, continuous control inputs $u \in \mathcal{U}$, discrete control inputs $\sigma \in \Sigma$, vector field $f : \mathfrak{Q} \times \mathfrak{X} \times \mathcal{U} \to \mathfrak{X}$, $(q_i, x, u) \mapsto A_ix + B_iu$, and transition function $R : \mathfrak{Q} \times \mathfrak{X} \times \mathcal{U} \times \Sigma \to \mathfrak{Q} \times \mathfrak{X}$.

Let $\mathcal{X}_f(q_i)$ (a set of continuous states in mode $q_i$) be the target set and $\mathcal{W}(q_i)$ the reachable set. Also, let $T_i$ be the transformation matrix for mode $q_i$ obtained from the complexity reduction approach described previously. As in [20], reachability calculations proceed in each mode in parallel such that for mode $q_i$ the reach-avoid operation becomes

$$T_i Reach\big(T_i^{-1}\mathcal{X}_f(q_i), T_i^{-1}\mathcal{W}(q_i)\big). \quad (39)$$

In case of a switched system with two modes $q_i$ and $q_j$ and an identity reset map, the backward reachable set $\mathcal{X}_t$ can be directly calculated as

$$\mathcal{X}_t = T_j Reach\Big(q_j, T_j^{-1}T_i Reach\big(q_i, T_i^{-1}\mathcal{X}_f(q_i)\big)\Big) \quad (40)$$

where $T_i$ and $T_j$ are the transformation matrices for modes $q_i$ and $q_j$ respectively. Reachability analysis is then performed on lower-dimensional subsystems in each mode according to Algorithm 1.

## IV. Numerical Example

Although the complexity reduction scheme presented here can be used in conjunction with any reachability technique, we demonstrate the applicability and practicality of our method using an example that employs the Level Set Toolbox (LS) [21]. While the LS has mainly been used for systems of low dimensionality, [22], our complexity reduction approach can facilitate its use for a class of higher dimensional systems for which safety controller synthesis and handling of non-convex or arbitrarily-shaped sets is important.

The following computations are performed on a dual core Intel-based machine with 2.8 GHz CPU, 6 MB of cache and 3 GB of RAM running single-threaded 32-bit MATLAB 7.5.

### A. 4D Aircraft Dynamics

Consider longitudinal aircraft dynamics $\dot{x} = Ax + B\delta_e$,

$$A = \begin{bmatrix} -0.0030 & 0.0390 & 0 & -0.3220 \\ -0.0650 & -0.3190 & 7.7400 & 0 \\ 0.0200 & -0.1010 & -0.4290 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0.0100 \\ -0.1800 \\ -1.1600 \\ 0 \end{bmatrix}$$

with state $x = [u, \alpha, \dot{\theta}, \theta]^T \in \mathbb{R}^4$ comprised of deviations in aircraft speed, angle of attack, pitch-rate, and pitch angle respectively, and with input $\delta_e \in [-13.3°, 13.3°] \subset \mathbb{R}$ the elevator deflection. These matrices represent stability derivatives of a Boeing 747 aircraft cruising at an altitude of 40 kft with speed 774 ft/sec [23]. We define a target (unsafe) set $\mathcal{X}_f$ such that in the transformed coordinate space $\mathcal{Y}_f = \{y \in \mathbb{R}^4 \mid \|y\| > 0.15,\ y = T^{-1}x,\ x \in \mathcal{X}_f\}$ where $T$ is the transformation matrix obtained through our method.

We decompose the system into two 2D subsystems. The reachability calculations are performed over a grid with 41 nodes in each dimension for $t_f = 3$ seconds. The computation time for the actual and the transformation-based reachable sets (including decomposition and projections) were 17352.0 and 33.5 seconds, respectively—a significant reduction.

Since the computed sets are 4D, we plot a series of 3D snapshots of these 4D objects at specific values of $y_4$ (Fig. 1). The aircraft flight envelope (safe) is represented by the area inside the shaded regions.

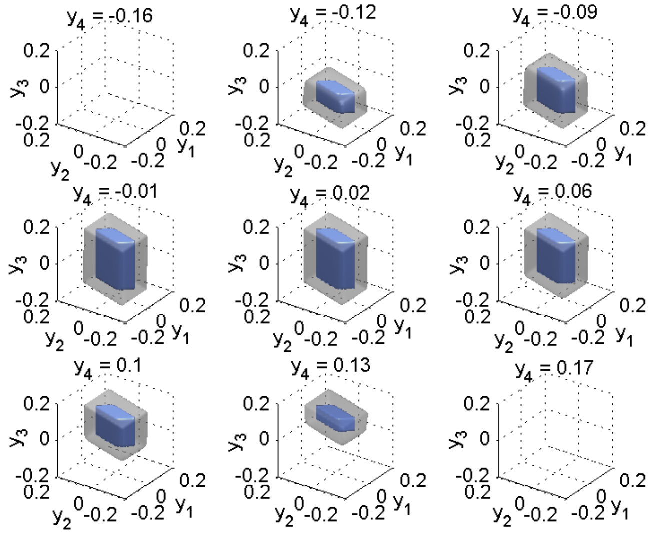


Fig. 1. Transformation-based (solid dark) vs. actual (transparent light) safe sets in the transformed coordinate space for the aircraft example (4D)

## V. Conclusions

In this paper we presented a complexity reduction technique based on structure decomposition for LTI dynamics. We showed that this decomposition has considerable potential for reducing the computational efforts in reachability analysis, especially for reachability tools that are computationally intensive (e.g. the Level Set Toolbox).

In comparison to the algorithmic approach presented in [14], we showed that using a one-time-operation (through a series of transformations) we can guarantee that the resulting subsystems are unidirectionally weakly coupled across which the input is disjoint. However, we observe a trade-off between the degree of coupling (which is directly proportional to the conservatism in the computation of reachable set) and the condition of the transformation matrix $T$. While weakening the coupling term, care must be taken so that the transformation does not distort the target set in the new coordinate space.

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