Second Order Cone Programming for Sensor Network Localization with Anchor Position Uncertainty

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Abstract
Node localization is a difficult task in sensor networks in which the ranging measurements are subject to errors and anchor positions are subject to uncertainty. In this paper, the robust localization problem is formulated using the maximum likelihood criterion under an unbounded uncertainty model for the anchor positions. To overcome the non-convexity of the resulting optimization problem, a convex relaxation leading to second order cone programming (SOCP) is devised. Furthermore, an analysis is performed in order to identify the set of nodes which are accurately positioned using robust SOCP, and to establish a relation between the solution of the proposed robust SOCP optimization and the existing robust optimization using semidefinite programming (SDP). Based on this analysis, a mixed robust SDP-SOCP localization framework is proposed which benefits from the better accuracy of SDP and the lower complexity of SOCP. Since the centralized optimization involves a high computational complexity in large networks, we also derive the distributed implementation of the proposed robust SOCP convex relaxation. Finally, we propose an iterative optimization based on the expectation maximization algorithm for the cases where anchor uncertainty parameters are unavailable. Simulations confirm that the robust SOCP and mixed robust SDP-SOCP provide tradeoffs between localization accuracy and computational complexity that render them attractive solutions, especially in networks with a large number of nodes.

Index Terms
Localization, Wireless sensor networks, Second order cone programming, Robust optimization, Distributed localization.

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I. INTRODUCTION

The availability of accurate information about the location of nodes is essential in many sensor network applications, for example target tracking and detection, cooperative sensing, and energy-efficient routing [2], [3]. A common approach for sensor localization is to utilize the (noisy) ranging information between sensor nodes and anchor nodes with \textit{a priori} known location in order to estimate the sensor positions. This ranging information is obtained for nodes which are in the communication range of each other by measuring received signal strength [4], [5], angle of arrival [6], or time of arrival (ToA) [7], [8]. For example, in ultra wideband (UWB) sensor networks, very accurate ranging information (with an accuracy of a few centimeters) is available using the ToA technique at no extra cost whenever two UWB sensors communicate with each other. These precise ranging measurements can then be passed to a localization algorithm in order to determine the location of sensors [9].

A number of approaches have been proposed to reduce the complexity of the localization problem in the general case, e.g. [10]–[16]. For example, the multi dimensional-scaling approach (MDS-MAP) by Shang \textit{et al.} [10] constructs an approximate map of the network based on the shortest-path distance information between the nodes, and then applies MDS to the map in order to find the relative locations of the nodes. The weighted least square method can be also used for obtaining approximate closed-form location estimates [16], but this method relies on the availability of a good initial estimate.

Another approach is to relax the original problem to obtain a convex optimization problem, which can be efficiently solved using existing algorithms such as interior point methods [17]. The two main convex relaxation techniques that have been considered for the sensor localization problem are second order cone programming (SOCP) [18], [19], and semidefinite programming (SDP) [20]–[22]. Authors in [20] use SDP to determine and localize the subset of nodes which are uniquely localizable in the absence of noise. It is shown that if the network is uniquely localizable, SDP can verify and provide the exact solution in a polynomial time. However, both SDP and SOCP provide suboptimal solutions in the presence of noise.

The comprehensive study by Tseng [19] investigates the SOCP and SDP formulations for the localization problem and studies several interesting properties, such as the relation of the SOCP and SDP solutions to the convex hull of anchors, conditions on uniqueness of the solutions,
characterizing the optimal objective values, and how to make the formulations distributed. Tseng further shows that there is a tradeoff between SDP and SOCP in terms of complexity and localization accuracy. While SDP provides a tighter relaxation and hence results in a better localization accuracy compared to SOCP, SOCP has a lower computational complexity and takes a shorter time to solve. The lower complexity of SOCP is because for a given problem, the number and size of variables and constraints required for solving SOCP are smaller than those required for solving SDP. This tradeoff motivates the design of mixed SDP-SOCP algorithms which benefit from the better accuracy of SDP and the lower complexity of SOCP.

Most sensor localization algorithms assume accurate anchor positions in order to estimate the location of the rest of the sensor nodes. However, in many scenarios anchor positions may not be accurately known. This uncertainty in turn significantly affects the quality of the estimated sensor positions. Robust sensor localization under anchor position uncertainty is studied in [23], where a convex relaxation based on SDP is developed. More specifically, a maximum likelihood criterion consisting of two parts is optimized. The first term reflects the likelihood of measurements, which is common in the robust and non-robust optimization. The second term is the likelihood of anchor positions, which forces the problem to find a reliable solution despite any errors in the initial measurement(s) of the anchor positions. We follow the model in [23] and provide an algorithm based on robust SOCP with a significantly lower computational complexity. In doing so, our work specifically extends [19], [20], [23] by exploring tradeoffs between the robust versions of SDP and SOCP.

Another important issue for the localization problem is the scalability to larger networks. In fact, when the number of nodes in the network is large, the centralized formulations of MDS-MAP, SDP and SOCP include many optimization variables and constraints, and therefore, solving them entails a high computational complexity. A three-phase distributed refinement algorithm is proposed for the SOCP [18] and a distributed MDS approach is proposed in [24] which uses the local ranging information in the sensors to calculate their locations. A distributed version of SDP is also proposed in [22], which can localize large networks by solving SDP on small clusters of nodes. If the anchors are not distributed uniformly, this method requires intelligent clustering to accurately localize nodes that are connected to only a few anchors (usually at the network boundaries). A different approach for reducing the complexity of SDP is known as the edge-based method [23], [25], [26], which further relaxes SDP by breaking a large constraint
into smaller ones at the cost of reduced localization accuracy. Edge-based SDP methods can be also solved in a distributed manner [27]. In this paper we provide a new distributed algorithm based on SOCP. Different from the distributed SDP algorithm proposed in [22], it does not require clustering. It is faster compared to the edge-based SDP methods [27], and also leads to a more accurate localization when compared to the existing distributed robust approaches [18].

In this paper, we derive the mathematical formulations for a robust SOCP relaxation of the localization problem under the unbounded anchor uncertainty model. Our motivation behind choosing robust SOCP is that it is significantly faster compared to robust SDP. We also show that the results of identifying the uniquely-localizable nodes in [19] hold for the developed SOCP-based method in the presence of anchor position uncertainties. Furthermore, we show the relation between robust SOCP and SDP, based on which we also generalize the idea of mixed SDP-SOCP from [19]. Quantitative results from simulations demonstrate the performance-complexity tradeoff between SDP, SOCP, and mixed SDP-SOCP. The proposed robust SOCP is further extended to a fully distributed method, which is not available for the solutions in [23] and has the advantage of being robust to anchor location uncertainties compared to the distributed algorithm in [18].

When formulating the robust localization schemes, we also consider the possibility of unknown covariance of the uncertainty distribution. For this case, we propose an expectation maximization (EM) method to iteratively estimate the covariance matrix of the uncertainty, which is original in the context of robust localization.

In summary, we present a set of novel methods for localization under anchor position uncertainty, namely robust SOCP, combined robust SOCP and SDP, a distributed robust SOCP solution, and an EM based method dealing with unknown parameters, which (i) complement the suite of methods available in literature, and (ii) have their distinct advantages over counterparts from literature in terms of computational complexity, scalability, and/or independence from unknown uncertainty parameters. These advantages are highlighted through extensive numerical results, which also demonstrate the ability of the proposed algorithms for accurate node localization under anchor uncertainty in large networks.

The rest of this paper is organized as follows. The SOCP formulation with unbounded anchor position uncertainty and its unambiguous localizability property are presented respectively in Sections II and III. We then show the relation between robust SOCP formulation and robust
SDP and propose a localization method based on mixing these two methods in Section IV. In Section V, the robust SOCP formulation is extended to its distributed form, and an EM method for the case of unknown uncertainty covariance is proposed. Simulation results to demonstrate the performance of the proposed methods are presented in Section VI, followed by conclusions in Section VII.

II. THE SENSOR LOCALIZATION PROBLEM

We consider a sensor network that consists of a set of anchors, \( \mathcal{N}_a \), and a set of general sensor nodes \( \mathcal{N}_s \). We denote \( \mathcal{N} = \mathcal{N}_s \cup \mathcal{N}_a \) as the set of all nodes in the network, and \( \mathbf{x}_i^0 \) as the vector that contains the actual coordinates of a node \( i \in \mathcal{N} \). Furthermore, we denote the number of anchors by \( k = |\mathcal{N}_a| \) and the total number of nodes by \( m = |\mathcal{N}| \). Although we assume two-dimensional coordinates in this paper, the extension to higher dimensions is straightforward.

The sensor network can be represented by an undirected graph \( \mathcal{G} = (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} \) is the set of all sensors as defined above, and \( \mathcal{E} \) is the set of links. A link \( (i, j) \in \mathcal{E} \) if nodes \( i, j \) are neighbors, i.e. they are in the communication range, \( R_c \), of each other: \( d_{ij}^0 = \| \mathbf{x}_i^0 - \mathbf{x}_j^0 \| \leq R_c \).

Similar to [23], we assume that \( \mathcal{E} \) does not include anchor-anchor links. Because of errors in the ranging measurements, the estimated distance between nodes \( i \) and \( j \) can be written as

\[
d_{ij} = d_{ij}^0 + e_{ij}.
\]

The measurement error \( e_{ij} \) is modeled as a zero-mean Gaussian random variable with variance \( \sigma_{ij}^2 \) [23]. We denote \( \mathcal{D} \) as the set of all available noisy measurements

\[
\mathcal{D} = \{ d_{ij} \mid (i, j) \in \mathcal{E} \}.
\]

A. Localization with Perfectly Known Anchor Positions

Let us first assume a sensor network in which the anchor positions are perfectly known. Given the anchor positions \( \mathbf{x}_i^0, i \in \mathcal{N}_a \), and the set of noisy measurements \( \mathcal{D} \), the goal of node localization is to obtain estimates of general sensor node positions, \( \mathbf{x}_i, i \in \mathcal{N}_s \), that minimize the sum of squared measurement errors based on the maximum likelihood criterion:

\[
\min_{\mathbf{x}_i, i \in \mathcal{N}_s} \sum_{(i,j) \in \mathcal{E}} \left( \| \mathbf{x}_i - \mathbf{x}_j \| - d_{ij} \right)^2.
\]
Since this optimization problem is non-convex and in its general form NP-hard to solve [28], convex SOCP and SDP relaxation techniques have been applied to (2) and its variants in [18], [19], [21]–[23], [27].

B. Localization with Anchor Position Uncertainty

In practice, perfect knowledge of anchor positions may not be available. In many scenarios, the anchor positions are obtained using global positioning system (GPS) and thus subject to estimation errors, which usually are modeled as Gaussian random variables [16], [23], [29]. Accordingly, the uncertain anchor positions are given by

\[ \mathbf{a}_i = \mathbf{x}_i^0 + \mathbf{w}_i, \quad i \in \mathcal{N}_a, \]  

(3)

where the position uncertainty of the \( i \)th anchor, \( \mathbf{w}_i \), is a zero-mean Gaussian random vector with covariance matrix \( \mathbf{\Psi}_i \).

Using the anchor position uncertainty model in (3), our goal is to obtain a robust counterpart of the localization problem in (2) that explicitly takes into account the uncertainty in the anchor positions. We note that the robust problem involves the refinement of the anchor positions. Following [23], we consider a maximum likelihood estimation (MLE) approach for \( \mathbf{x}_i^0, \ i \in \mathcal{N} \), and examine the probability

\[ P(D, \{\mathbf{a}_i\} | \{\mathbf{x}_i\}) = P(D | \{\mathbf{x}_i\}) \times P(\{\mathbf{a}_i\} | \{\mathbf{x}_i\}) \]

\[ = \prod_{(i,j) \in \mathcal{E}} \frac{1}{\sqrt{2\pi} \sigma_{ij}} \exp \left( -\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2}{2\sigma_{ij}^2} \right) \]

\[ \times \prod_{i \in \mathcal{N}_a} \frac{1}{\sqrt{2\pi} \mathcal{N} \mathbf{\Psi}_i} \exp \left( -\frac{(\mathbf{a}_i - \mathbf{x}_i)^T \mathbf{\Psi}_i^{-1} (\mathbf{a}_i - \mathbf{x}_i)}{2} \right). \]  

(4)

By taking the logarithm of (4), the robust localization problem can be written as

\[ \min_{\mathbf{x}_i, i \in \mathcal{N}} \sum_{(i,j) \in \mathcal{E}} \frac{(\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij})}{\sigma_{ij}^2} + \sum_{i \in \mathcal{N}_a} (\mathbf{a}_i - \mathbf{x}_i)^T \mathbf{\Psi}_i^{-1} (\mathbf{a}_i - \mathbf{x}_i). \]  

(5)

Similar to the optimization problem in (2), the robust localization problem (5) is non-convex. To obtain an SOCP relaxation of (5), we first write the optimization problem in the following
equivalent form:

$$\min_{\{x_i\}, \{t_{ij}\}, \{s_i\}} \sum_{(i,j) \in \mathcal{E}} t_{ij}^2 + \sum_{i \in \mathcal{N}_a} s_i^2$$

subject to

$$g_{ij} ||x_i - x_j|| - d_{ij} \leq t_{ij}, \quad (i, j) \in \mathcal{E}, \quad (6b)$$

$$\|\Psi^{-1/2}_i(a_i - x_i)\| \leq s_i, \quad i \in \mathcal{N}_a, \quad (6c)$$

where $g_{ij} \doteq \frac{1}{\sigma_{ij}}$. By defining a vector $u$ as the concatenation of variables in the objective, i.e.,

$$u \doteq [t_{ij} \ (i, j) \in \mathcal{E}, \ s_i \ i \in \mathcal{N}_a],$$

we can write (6) as the following equivalent epigraph form:

$$v_{socp} \doteq \min_{\{x_i\}, \{u\}, \{q_{ij}\}, v} \quad v$$

subject to

$$\|u\| \leq v, \quad (7b)$$

$$g_{ij} |q_{ij} - d_{ij}| \leq t_{ij}, \quad (i, j) \in \mathcal{E}, \quad (7c)$$

$$\|\Psi^{-1/2}_i(a_i - x_i)\| \leq s_i, \quad i \in \mathcal{N}_a, \quad (7d)$$

$$\|x_i - x_j\| = q_{ij}, \quad (i, j) \in \mathcal{E}. \quad (7e)$$

Finally, a convex SOCP relaxation of the robust localization problem in (7) can be obtained by relaxing the equality constraint (7e) to inequality

$$\|x_i - x_j\| \leq q_{ij}, \quad (i, j) \in \mathcal{E}. \quad (8)$$

Substituting (7e) with (8) results in a standard SOCP program which can be solved using an interior point algorithm [17].

In the special case of perfectly known anchor positions, i.e. $x_i = a_i = x_i^0, \ i \in \mathcal{N}_a$, (7) reduces\(^1\) to the standard SOCP formulation in [19]. The bounded uncertainty model in [27] is another special case of (7) by setting $\Psi_i = I_2$ and $s_i = \delta$, where $\delta$ is the given uncertainty bound. Also note that SOCP location estimates always lie within the convex hull of the anchor positions, which is not in general true for the SDP solution [19]. It is also worth mentioning that the sensors in (7) can be viewed as anchors with no prior information.

\(^1\)We should point out that the objective function of SOCP relaxation in (5) follows the MLE approach [23] and is slightly different from that of [27].
III. Unambiguously Localizable Nodes

Having formulated the localization problem as a robust SOCP (7), in this section we try to characterize the solution and establish a structure that shows how well it performs on each node. In fact, an advantage of using robust SOCP to localize the network is that it can provide necessary and sufficient conditions for identifying the set of uniquely localizable nodes with no additional cost [19]. That is, once the robust SOCP (7) is solved, we can immediately separate the set of nodes which have been accurately and uniquely localized\(^2\).

A node \(i\) is called unambiguously localizable if the minimum value of (7) is achieved for only a unique value of \(x_i\). In other words, unambiguous localizability of a sensor node \(i\) means that its estimated location is invariant over all solutions, which is in fact a property of the convex optimization that is being used. Note that this definition is different from the definition of “unique localizability” from the viewpoint of network rigidity in the absence of noise, for example in [28], [30]. Here, the term unambiguously localizable refers to the set of nodes whose estimated locations are invariant over all solutions of the robust SOCP problem (7) [19], [20].

Let \(B = \{(i, j) \in E \text{ s.t. } \|x_i - x_j\| = q_{ij}, \}\) be the set of links which satisfy the SOCP relaxed constraint (8) with equality for every relative interior solution\(^3\) to the robust SOCP problem. We hereafter refer to the links in \(B\) as tight links, and to the corresponding nodes as tightly-connected nodes. The set of tightly-connected nodes is denoted by \(M\).

Tight links are essentially indicators of less uncertainty in the optimization problem, since the optimization has found a unique optimal value for their corresponding nodes. In fact, Tseng [19] shows that the tightly-connected nodes are accurately localized in the sense that their localization error is less than the square root of their average link measurement errors. Furthermore, it is mathematically proved that, for the case of perfectly known anchor positions, the set of unambiguously-localizable nodes are exactly those which are tightly-connected. In other words, the optimization solution for not tightly-connected nodes is not unique. In these situations, the

\(^2\)Note that having non-unique solutions for some nodes in a convex relaxation is common in the context of network localization [19], [21] when measurement noise is present. That is, any solution is as good as any other among the set of obtained non-unique solutions, as far as the optimization objective is concerned. If desired, a specific solution can be forced through regularization methods [17, Section 6.3]

\(^3\)A relative interior solution lies inside the relative interior of the solution set. In our case, all solutions which satisfy the remaining constraints corresponding to \(E - B\) with strict inequality are relative interior solutions.
localization algorithm can return any of the solutions, for example (and as in our simulations) the analytic center solution which corresponds to the center of the solution set. Mathematically, the product of all slack variables corresponding to non-tight constraints is maximized at the analytic center solution. Geometrically, the analytic center solution has the maximum “distance” to the solution set boundaries. The nodes which are not tightly-connected, and hence their location estimates are not unique, can be further re-localized using the combined method presented in Section IV-B for more accuracy.

In order to extend the above-mentioned relation between uniqueness of the solution and tightly-connected nodes to the robust case, we note that in the robust SOCP the \(i\)th anchor position can vary in an area around \(a_i\) and the proof of [19] for unambiguous localization does not directly apply. In fact, here we first need to establish that all of the anchors are unambiguously localizable.

**Proposition 1:**

a) All anchors are unambiguously localizable in (7).
b) A sensor node \(i\) is unambiguously localizable in (7) if and only if it is tightly-connected, i.e., \(x_i\) is invariant over all solutions \(\Leftrightarrow i \in M\).
c) At the analytic center solution, the estimated location of all nodes lie inside the convex hull of the estimated anchor locations, i.e., all nodes on the convex hull of the analytic center solution are anchors.

**Proof:** See Appendix A.

Proposition 1a) states that solving (7) leads to an unambiguous location for anchors. This is due to the availability of a prior on the anchor positions. In fact, if we also had a prior of the same form for sensors, \(\Psi_i^{-1/2}(a_i - x_i), \ i \in N_s\), then all sensors would have been unambiguously localizable as well. This property follows from the fact that the point at which the contour defined by the convex prior touches the convex solution set of \(x_i, \ i \in N_s\), is unique [17]. In other words, adding the information \(a_i, \Psi_i, \ i \in N_s\) to the problem leads to the selection of a unique solution for \(x_i, \ i \in N_s\), which has the smallest Mahalanobis distance\(^4\) to \(a_i\), with regard to \(\Psi_i\). Part b) shows that the the set of tight nodes in the robust SOCP have similar properties to those in the original SOCP. An important corollary from combining parts b) and c) is that the sensor nodes that lie outside the convex hull of the anchors are not tightly-connected. Moreover, part c) provides a useful characteristic for the analytic center solution of robust SOCP which is not

\(^4\)The Mahalanobis distance of \(x_i\) from \(a_i\) with regard to the covariance matrix \(\Psi_i\) is defined as \(\|\Psi_i^{-1/2}(a_i - x_i)\|\).
in general true for its SDP counterpart.

IV. CONNECTION BETWEEN ROBUST SOCP AND SDP LOCALIZATION PROBLEMS

In this section, we first analyze the relation between (7) and the existing robust SDP formulation [23] in Section IV-A, and then devise an optimization problem for mixing the robust SOCP and SDP in Section IV-B which enables a flexible tradeoff between accuracy and complexity.

A. Relation of Robust SOCP and Robust SDP Relaxations

For the case of standard SOCP and SDP, Tseng [19] shows that the set of all possible solutions obtained from SDP relaxation is a subset of all possible solutions that can be obtained by SOCP. In other words, the SOCP solution set includes the SDP solution set, and hence SDP provides a tighter relaxation. In this section we show a similar relation between the robust formulation of SOCP (7) and the robust SDP in [23].

Introducing $X$ as a $2 \times m$ matrix whose $i^{th}$ column is set to $x_i$, $Y = \begin{bmatrix} X^T X & X^T \end{bmatrix}$, $\gamma_{ij} = \|x_i - x_j\|^2$, and $\Xi_i = x_i x_i^T$, and applying the SDP relaxations, [23] obtains the following robust SDP formulation:

$$v_{\text{sdp}} = \min_{X, Y, \{\Xi_i\}, \{\gamma_{ij}\}, \{r_{ij}\}} \sum_{(i,j) \in E} g_{ij}^2 (\gamma_{ij} - 2d_{ij} r_{ij}) + \sum_{i \in N_a} (\text{tr} (\Psi_i^{-1} \Xi_i) - 2a_i^T \Psi_i^{-1} x_i) \tag{9a}$$

subject to

$$\gamma_{ij} = y_{ii} + y_{jj} - y_{ij} - y_{ji}, \quad (i, j) \in E, \tag{9b}$$

$$r_{ij}^2 \leq \gamma_{ij}, \quad (i, j) \in E, \tag{9c}$$

$$\text{tr} (\Xi_i) = y_{ii}, \quad i \in N_a, \tag{9d}$$

$$x_i = \begin{bmatrix} y_{im+1} & y_{im+2} \end{bmatrix}^T, \quad i \in N, \tag{9e}$$

$$\begin{bmatrix} y_{m+1m+1} & y_{m+1m+2} \\ y_{m+2m+1} & y_{m+2m+2} \end{bmatrix} = I_2, \tag{9f}$$

$$\begin{bmatrix} \Xi_i & x_i \\ x_i^T & 1 \end{bmatrix} \succeq 0_3, \quad i \in N_a, \tag{9g}$$

$$Y \succeq 0_{m+2}, \tag{9h}$$

where $\text{tr}(\cdot)$ is the trace of a matrix, \( \succeq 0_n \) stands for positive semidefiniteness of an $n \times n$ matrix, and $y_{ij}$ denotes the element in the $i^{th}$ row and $j^{th}$ column of $Y$. The objective (9a) is
obtained by removing the constant term

\[ v_0 = \sum_{(i,j) \in \mathcal{E}} g_{ij}^2d_{ij}^2 + \sum_{i \in \mathcal{N}_a} a_i^T \Psi_i^{-1}a_i, \quad (10) \]

from the log-likelihood (5). The following proposition, whose proof is provided in Appendix B, shows the relation between robust SOCP formulation (7) and robust SDP (9).

**Proposition 2:** Let \( S_{\text{sdp}} = \{X, Y, \{\Xi_i\}, \{\gamma_{ij}\}, \{r_{ij}\}\} \) be a feasible solution for robust SDP constraints (9b)-(9h). Then, the set of variables \( S_{\text{socp}} = \{\{x'_i\}, \{q_{ij}\}, \{t_{ij}\}, \{s_i\}, v\} \) defined as

\[
\begin{align*}
x'_i &= x_i, \\
g_{ij} &= \sqrt{\gamma_{ij}}, \quad (i, j) \in \mathcal{E}, \\
t_{ij} &= g_{ij}\left|g_{ij} - d_{ij}\right| = g_{ij}\left|\sqrt{\gamma_{ij}} - d_{ij}\right|, \quad (i, j) \in \mathcal{E}, \\
M_i &= \Xi_i + a_i a_i^T - a_i x_i^T - x_i a_i^T, \quad i \in \mathcal{N}_a, \\
s_i &= \left(\text{tr}\left(\Psi_i^{-1} M_i\right)\right)^{\frac{1}{2}}, \quad i \in \mathcal{N}_a, \\
v &= \left(\sum_{(i,j) \in \mathcal{E}} t_{ij}^2 + \sum_{i \in \mathcal{N}_a} s_i^2\right)^{\frac{1}{2}},
\end{align*}
\]

forms a feasible solution for the robust SOCP constraints (7b)-(7d) and (8). Furthermore, there exists a robust SDP formulation equivalent to (9) whose optimum value is \( v'_{\text{sdp}} = \sqrt{v_{\text{sdp}}} + v_0 \)

and this adjusted optimum value is always greater than or equal to the robust SOCPs optimum value (7a), i.e.,

\[ v_{\text{socp}} \leq \sqrt{v_{\text{sdp}}} + v_0, \quad (12) \]

where \( v_0 \) is defined in (10).

**Proof:** See Appendix B.

Proposition 2 states that the robust SDP solution set is contained in the robust SOCP solution set if (12) holds with equality. Note that the equivalent SDP formulation in Proposition 2 is obtained by applying the mapping \( \Xi_i \rightarrow M_i \) in (11) to (9), as explained in Appendix B. The main importance of this proposition is that it extends the results of [19] for the non-robust case to the robust case. In general, robust SDP provides a tighter relaxation for the localization problem compared to robust SOCP.

We close this section by noting that the authors in [23] have calculated the Cramer-Rao lower bound (CRLB) for robust localization problem. Since the CRLB is independent of the
solution being used and we have the same noise model as [23], the same formulation can be used here. Specifically, the CRLBs for the location variables are given by the corresponding diagonal elements of the inverse of the Fisher information matrix

$$F_{(2m \times 2m)} = H_a \text{ blkdiag}^{-1}(\{\Psi_i, \ i \in \mathcal{N}_a\}) H_a^T + H_s \text{ diag}^{-1}(\{\sigma_{ij}^2, \ (i, j) \in \mathcal{E}\}) H_s^T,$$

(13)

where $H_a, (2m \times 2k) = [I_{(2k \times 2k)} 0_{(2m-2k \times 2k)}]^T$, and

$$H_s, (2m \times |\mathcal{E}|) = \text{cat}(\{[0 \ldots (x^o_i - x^o_j)^T/d^o_{ij} \ldots 0 \ldots (x^o_i - x^o_j)^T/d^o_{ij} \ldots 0]^T, \ (i, j) \in \mathcal{E}\}),$$

where each column contains only 2 non-zero elements corresponding to an edge, and $0, \text{ diag}(.)$, blkdiag$(.)$, and cat$(.)$ denote all-zero, diagonal, block diagonal, and column-wise-concatenated matrices, respectively.

B. Combination of Robust SOCP and SDP

As suggested by Tseng [19], the SOCP and SDP problems can be mixed in order to provide an accuracy-complexity tradeoff for the localization problem. Such a combination is possible by dividing the set of nodes into two sets $\mathcal{N}_{socp}$ and $\mathcal{N}_{sdp}$ and running SOCP and SDP for these sets with the corresponding ranging information, respectively. Specifically, we have

$$\min_{x, y, \{\xi_i\}, \{\gamma_{ij}\}, \{w_{ij}\}, \{v_{ij}\}, v} \sum_{(i,j) \in \mathcal{E}_{sdp}} \gamma_{ij}^2 (\gamma_{ij} - 2d_{ij}r_{ij}) + \sum_{i \in \mathcal{N}_{a,sdp}} \text{tr} (\Psi_i^{-1}\xi_i) - 2\alpha_i^T\Psi_i^{-1}x_i) + v \quad (14)$$

subject to

(9b)-(9h) for $\mathcal{N}_{sdp}, \mathcal{E}_{sdp}, D_{sdp}$,

(7b)-(7d) and (8) for $\mathcal{N}_{socp}, \mathcal{E}_{socp}, D_{socp},$

where subscripts ‘sdp’ and ‘socp’ on $\mathcal{N}, \mathcal{E}, \mathcal{D}$ indicate the sets of nodes, links and ranging measurements belonging to SDP and SOCP sub problems, respectively. Note that in general there is no reason to force completely separate problem sets, i.e. one can have $\mathcal{N}_{socp} \cap \mathcal{N}_{sdp} \neq \emptyset$.

The main task here is to efficiently assign the nodes to these two problem sets so that the overall performance, which can be a function of both computational complexity and localization accuracy, is optimized. Here we propose a simple hybrid algorithm which is faster than robust SDP and provides a better accuracy compared to robust SOCP.

- Step 1: Solve robust SOCP problem (7) for the set of all sensors and anchors $\mathcal{N}_{socp} = \mathcal{N}$. 

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• Step 2: Let $B$ be the set of tight links in the SOCP solution as defined in Section III. Then, construct $\mathcal{N}_{\text{sdp}} = \{i \text{ s. t. } \forall (i, j) \in \mathcal{E} : (i, j) \notin B\} \cup \mathcal{N}_a$. In other words, the robust SDP is solved over the subset of nodes without tight links, as well as anchors.

Note that the method in (14) is more general than the idea proposed in [19], in the sense that we allow combined optimizations where $\mathcal{N}_{\text{sdp}} \cap \mathcal{N}_{\text{socp}} \neq \emptyset$. In fact, what is novel about the proposed combined method is that we use the SOCP result over $\mathcal{N}_{\text{socp}} = \mathcal{N}$ to determine $\mathcal{N}_{\text{sdp}}$. Obviously, the complexity of this combined method is higher than that of the robust SOCP. However, the robust SDP step typically needs to be solved over the subset of non-tight nodes that usually consists of 10-20% of total nodes in the network [19], and therefore the complexity of the proposed combined method is still significantly lower compared to that of a robust SDP solved individually over the set of all nodes. We should also mention that for sparse networks, a subset of tight nodes can be kept for step 2 to avoid graph disconnectivity [31]. This technique is however not used in our simulations for the combination algorithm since we have considered sufficiently dense topologies.

V. EXTENSIONS OF THE SOC FORMULATION

In this section we provide different extensions to the SOC formulation in (7) We first give a distributed version of the problem (Section V-A) and compare the complexity of different centralized and distributed localization methods (Section V-B). Then, we consider the case that the covariance matrices of the uncertainty model, $\Psi_i$, are unknown, and propose an EM algorithm to iteratively estimate these parameters (Section V-C). Finally, we mention a gradient-based refinement algorithm which can be applied to the results obtained by the robust SOCP or SDP methods in order to reduce their positioning error (Section V-D).

A. DISTRIBUTED SOC FORMULATION

Another attractive feature of the SOC relaxation for the robust localization problem is its potential for distributed implementation, which allows the optimization problem to be divided into a number of smaller sub problems that can be solved locally at each node. These sub problems will depend on the position of the neighboring nodes. Hence, after the step of solving the sub problems locally, the nodes exchange the estimates of their positions with neighbor nodes.
A distributed algorithm was proposed for network localization with perfectly known anchor positions in [18]. In this case, the local sub problems are solved only at the general sensor nodes and not at the anchors. We generalize this distributed algorithm to the robust localization problem with uncertain anchor positions. Consider the robust localization problem in (5). It can be written as

\[
\begin{align*}
\min_{x_i, i \in \mathcal{N}} & \sum_{(i,j) \in \mathcal{E}} g_{ij}^2 (q_{ij} - d_{ij})^2 + \sum_{i \in \mathcal{N}_a} (a_i - x_i)^T \Psi_i^{-1}(a_i - x_i) \\
\text{subject to} & \|x_i - x_j\| = q_{ij}, \quad (i, j) \in \mathcal{E},
\end{align*}
\]

where constraint (15b) can be replaced by an inequality constraint for the SOCP relaxation. Using the barrier function approach, cf. [18], [27], [32], the relaxed constrained problem can be approximated as the following unconstrained problem:

\[
\begin{align*}
\min_{x_i, i \in \mathcal{N}, q_{ij}, (i,j) \in \mathcal{E}} & \sum_{(i,j) \in \mathcal{E}} g_{ij}^2 (q_{ij} - d_{ij})^2 + \sum_{i \in \mathcal{N}_a} (a_i - x_i)^T \Psi_i^{-1}(a_i - x_i) + \sum_{(i,j) \in \mathcal{E}} B(\|x_i - x_j\|^2 - q_{ij}^2), \\
\end{align*}
\]

where \(B(.)\) is a properly chosen barrier function, such as the logarithmic barrier function \(B(z) = -\frac{1}{c} \log(-z)\) for a large constant \(c \gg 1\). Now, the problem in (16) is partially separable and each term in the summation can be minimized independently at each node \(i\) using only information about the positions \(x_j\) and ranging information \(d_{ij}\) of the set of its neighbor nodes, \(\mathcal{K}_i\) [18]. Using an approach similar to the one used in the previous section, we can formulate the local sub problems to be solved at each anchor or general sensor node. Specifically, for each general node \(i \in \mathcal{N}_s\), the local sub problem can be approximated by iteratively solving

\[
\begin{align*}
\min_{x_i, t_{ij}, q_{ij}, v_i} & v_i \\
\text{subject to} & \|u_{t,i}\| \leq v_i, \quad (17b) \\
& g_{ij} |q_{ij} - d_{ij}| \leq t_{ij}, \quad j \in \mathcal{K}_i, \quad (17c) \\
& \|x_i - x_j\| \leq q_{ij}, \quad j \in \mathcal{K}_i, \quad (17d)
\end{align*}
\]

\(^5\)In general, there is no analytical proof that iteratively solving (17), (18) converges to the optimal solution of (7) [33], but this is usually the case in the simulations.
where \( \mathbf{u}_{t,i} = [t_{ij} | j \in \mathcal{K}_i] \). Moreover, for each anchor node \( i \in \mathcal{N}_a \), the local sub problem can be approximated by solving

\[
\min_{x_i, t_{ij}, q_{ij}, s_i, v_i} \quad v_i \\
\text{subject to} \quad \| \mathbf{u}_{t,s,i} \| \leq v_i, \tag{18a}
\]

\[
\| \Psi_i^{-1/2} (a_i - x_i) \| \leq s_i, \tag{18b}
\]

\[
g_{ij} |q_{ij} - d_{ij}| \leq t_{ij}, \quad j \in \mathcal{K}_i, \tag{18c}
\]

\[
\| x_i - x_j \| \leq q_{ij}, \quad j \in \mathcal{K}_i, \tag{18d}
\]

where \( \mathbf{u}_{t,s,i} = [t_{ij} | j \in \mathcal{K}_i, s_i] \). We observe that the local sub problem of each anchor is different from that of a general sensor node due to the robust formulation of the problem. However, these two formulations can be unified in the same way as the centralized version, i.e. by adding priors for sensors.

In an attempt to treat the case of uncertain anchor positions, the distributed algorithm for the case of known anchor positions in [18] suggested including the anchors in solving local sub problems. In their method, the sensors first perform a local SOCP phase to find their locations assuming perfect anchor positions. Then, the anchors use these estimations to refine their location information. In the third phase, the general sensors run a new round of local SOCP based on the refined location of anchors. However, the sub problems at sensors are run without taking into account the anchor uncertainty, and the sub problems at anchors are run without taking into account the prior information. Hence, a better performance of the proposed distributed robust approach is expected, as will be demonstrated in Section VI.

As the final remark, we should note that the combined robust SOCP and SDP in Section IV-B can be made distributed. In fact, the SOCP part of the proposed hybrid method (14) can be solved in a distributed manner using (17) and (18) above, and the SDP part can be also solved by following the existing distributed SDP methods e.g. [22]. However, distributed SDP would require clustering for a good accuracy and can still have a higher complexity in a highly-connected network. Hence, we focus on the distributed robust SOCP in our simulations and provide results only for the centralized combined method. When more accuracy is desired for the boundary nodes, the proposed distributed SOCP may be replaced by the distributed SDP with appropriate clustering methods and/or heuristic anchor refinements at the cost of a higher


### TABLE I

**COMPLEXITY OF ROBUST AND NON-ROBUST SDP AND SOCP.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Variables</th>
<th>Equality constraints and their size</th>
<th>Inequality constraints and their size</th>
<th>Total Constraints (Number × Size)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard SOCP [19]</td>
<td>$2</td>
<td>E</td>
<td>+</td>
<td>N</td>
</tr>
<tr>
<td>Robust SOCP (7)</td>
<td>$2</td>
<td>E</td>
<td>+</td>
<td>N</td>
</tr>
<tr>
<td>Distributed Robust SOCP Sensors (17)</td>
<td>$2</td>
<td>K_i</td>
<td>+ 2$</td>
<td>0</td>
</tr>
<tr>
<td>Anchors (18)</td>
<td>$2</td>
<td>K_i</td>
<td>+ 3$</td>
<td>0</td>
</tr>
<tr>
<td>Distributed Robust ESOC [27], each node</td>
<td>$2</td>
<td>K_i</td>
<td>+ 3$</td>
<td>0</td>
</tr>
<tr>
<td>Standard SDP [21]</td>
<td>$2</td>
<td>E</td>
<td>+ (</td>
<td>N</td>
</tr>
<tr>
<td>Robust SDP (9) [23]</td>
<td>$2</td>
<td>E</td>
<td>+ (</td>
<td>N</td>
</tr>
<tr>
<td>Robust ESDP [23]</td>
<td>$2</td>
<td>E</td>
<td>+ (</td>
<td>N</td>
</tr>
</tbody>
</table>

B. **Complexity Analysis**

Table I shows the number of variables and constraints for robust and non-robust versions of SDP and SOCP in terms of number of optimization variables, and equality and inequality constraints. Also, the number of variables and constraints for the robust edge-based SDP (ESDP) [23] and SOCP (ESOC) [27] are shown. Since the complexity is an increasing function of both number and size of the constraints, for an easier comparison we also show a combined metric for constraints which is the product of their size and their number in the last column. In this table, $|K_i|$ denotes the number of neighbours of node $i$, and $|E|$ is the number of links in the graph.

As can be seen from Table I, the robust SDP approach (9) which considers anchor uncertainty [23], has $2|E| + (|N| + 2)^2 + 4|N_a| + |N|$ variables, while the proposed robust SOCP
in (7) has only \(2|\mathcal{E}| + |\mathcal{N}| + |\mathcal{N}_a| + 1\) variables. Moreover, the constraint metric is smaller for the proposed robust SOCP. Since typically we have \(|\mathcal{E}| = \Omega(|\mathcal{N}|)\) [19], SOCP entails a computational complexity which grows linearly with the number of nodes (\(\Omega(|\mathcal{N}|)\)) compared to a quadratic growth (\(\Omega(|\mathcal{N}|^2)\)) for SDP [19]. This enables SOCP to solve large problems with e.g. \(|\mathcal{N}| > 200\) nodes. These numbers of variables and constraints also show that when the network is sparse, the complexity is significantly lower for RSOCP compared to RSDP. In the other extreme case of a highly dense network, i.e. \(|\mathcal{E}| = \Omega(|\mathcal{N}|^2)\), the number of RSOCP variables become asymptotically similar to those of RSDP, but the constraint metric still remains smaller by an asymptotic factor of 6 to 7.

It can be also observed from Table I that the distributed SOCP (17) reduces the number of variables and constraints to the order of the number of neighbouring nodes (node connectivity). Specifically, we have \(|\mathcal{N}_s|\) distributed general node sub problems (17) with \(2|\mathcal{K}_i| + 2\) variables and the constraint metric of \(6|\mathcal{K}_i| + 1\); and \(|\mathcal{N}_a|\) anchor sub problems (18) with \(2|\mathcal{K}_i| + 3\) variables and the constraint metric of \(6|\mathcal{K}_i| + 3\). Since these small sub problems are solved in parallel at different nodes, the distributed robust SOCP is scalable to the network size. This fact also holds for the robust ESOCP [27], which has almost similar number of variables and constraint metric to our method. We also observe that the robust SOCP has a smaller number of variables and constraint metric compared to the robust ESDP.

C. An Expectation Maximization Approach for Gaussian Uncertainty with Unknown Covariance

We have formulated the robust SOCP (7) for anchor position uncertainties assuming known covariances, \(\Psi_i, i \in \mathcal{N}_a\). In practice, however, the sensors’ knowledge about the anchor position errors may be limited, and thus the covariance matrices might be unknown. In this case, given \(n_m \geq 2\) measurements \(\{\alpha_{i,1}, \ldots, \alpha_{i,n_m}\}\) are available, we can iteratively estimate \(\Psi_i\) by the EM method as explained in the following. The \(n_m\) measurements can be obtained prior to the start of the robust localization by inquiring the underlying anchor location provider, e.g. GPS, cellular towers or buoys [23].

The log-likelihood for the covariance matrix \(\Psi_i\) given the location information can be written as

\[
L = \log P(\Psi_i | \{\alpha_{i,n}\}, \mathbf{x}_i) = -\frac{n_m}{2} \log \left(2\pi \left| \Psi_i^{-1} \right| \right) - \frac{1}{2} \sum_{n=1}^{n_m} (\alpha_{i,n} - \mathbf{x}_i)^T \Psi_i^{-1} (\alpha_{i,n} - \mathbf{x}_i),
\]
Given \( a_{i,1}, \ldots, a_{i,n_m} \)
and
\[
x^{(1)}_i = \frac{1}{n_m} \sum_{j=1}^{n_m} a_{i,j}
\]

M step
\[
\Psi_i^{(t+1)} = \frac{1}{n_m} \sum_{n=1}^{n_m} (a_{i,n} - x_i^{(t)}) (a_{i,n} - x_i^{(t)})^T
\]

E step
Find \( x^{(t+1)}_i \) from robust SOCP
with \( x^{(1)}_i \) and \( \Psi_i^{(t)} \)

Fig. 1. A schematic of the proposed EM algorithm. In each E step, a robust SOCP is executed and then the covariance is estimated in the M step.

where \( \left\{ \Psi_i \right\}_{i \in \mathcal{N}_a} \) are the parameters to be estimated. In the EM algorithm, the complete log-likelihood is first computed for a given set of observations (E step), and then the parameters are maximized based on this expression (M step). These steps are iteratively executed until they converge to a local optimum point for the likelihood.

**The E step:** Let \( \Psi^\ell_i \) be the estimated covariance matrix of \( w_i \) at iteration \( \ell \). The expectation of the likelihood for anchor \( i \in \mathcal{N}_a \) is given by

\[
Q_i = \mathbb{E}_{x_i | \left\{ a_{i,n} \right\}, \Psi^\ell_i} \{ L \} = -\frac{n_m}{2} \log \left( 2\pi \left| \Psi_i^\ell \right|^{-1} \right) - \frac{1}{2} \mathbb{E}_{x_i | \left\{ a_{i,n} \right\}, \Psi^\ell_i} \left( \sum_{n=1}^{n_m} (a_{i,n} - x_i)^T \Psi_i^\ell \left( \Psi_i^\ell \right)^{-1} (a_{i,n} - x_i) \right),
\]

where expectation is taken over the location variables. In order to calculate the expectation in (20), a probability distribution of \( x_i \) given \( a_{i,n} \) and \( \Psi^\ell_i \) is needed. However, such a distribution might in general be difficult to find since in our problem \( x_i \) also depends on the distance measurements in \( \mathcal{D} \). In order to simplify the problem, we propose to compute the expected location of \( x_i \) for given \( \Psi^\ell_i \) using the suboptimal robust SOCP formulation (7).

**The M step:** The maximization step involves taking partial derivative of \( Q_i \) with respect to the parameters \( \Psi^\ell_i \) and finding the corresponding root. Since the observations are i.i.d. from a Gaussian distribution, it is easy to show that the MLE of \( \Psi_i \) is the sample covariance. Hence,

\[
\Psi_i^{t+1} = \frac{1}{n_m} \sum_{n=1}^{n_m} (a_{i,n} - x_i) (a_{i,n} - x_i)^T,
\]
where $x_i$ is the expected location of $a_i$ which is obtained from the robust SOCP algorithm in the E step.

We stop the EM algorithm at iteration $\ell_0$ when $L^{\ell_0+1} - L^{\ell_0} < \epsilon = 0.1$, where the log-likelihood $L$ for each EM iteration is obtained from (19). Furthermore, the EM complexity can be approximated as $O(\ell_0 \text{ O(RSOCP)})$, where O(RSOCP) is the RSOCP complexity from Table I. Figure 1 shows the flowchart of the EM algorithm. As can be seen, the robust SOCP iteratively improves the anchor position estimations in the E step, which is utilized in the M step (21) for finding a better approximation for $\Psi_i$. The iterations continue until the values of $Q_i$ converge to a local optimum point. As it is typical in EM, random restart points for $\Psi_i^0$ can be considered to find different local optima and choose the best. We will use the sample covariance of measurements as the initial value of $\Psi_i^0$. It is interesting to note that the proposed EM algorithm can be also viewed as an offline decision feedback method for estimating the covariance [34].

D. Gradient Descent Refinement Method

It is worth pointing out that an iterative gradient-descent (GD) method can be employed in order to further refine the results obtained by the robust SDP or SOCP convex optimizations [21, Sec. V]. In each iteration $\ell$ of the GD method, the location of a sensor node is updated\(^6\) by moving in the opposite direction of the gradient. Specifically, let

$$f(x_i^\ell) = \sum_{j \in K_i} g_{ij}^2 (\|x_i^\ell - x_j^\ell\| - d_{ij})^2$$

be the local value of objective for sensor $i$, i.e. the summation of terms in (5) which depend on $x_i$, at iteration $\ell$. Then,

$$x_i^{\ell+1} = x_i^\ell - \alpha \frac{\partial f(x_i^\ell)}{\partial x_i^\ell}, \quad i \in N_s$$

where $\alpha$ is the step size, and the gradient is given by,

$$\frac{\partial f(x_i^\ell)}{\partial x_i^\ell} = \sum_{j \in K_i} 2g_{ij}^2 (x_i^\ell - x_j^\ell)(1 - \frac{d_{ij}}{\|x_i^\ell - x_j^\ell\|}).$$

It is shown in [21] that GD can refine the location of sensor nodes found by the convex optimization. However, GD fails to provide a good solution if a good initial solution is unavailable.

\(^6\)For simplicity, anchors are excluded from the GD refinement.
VI. PERFORMANCE EVALUATION

In this section we examine the performance of the proposed localization methods by means of simulations. We use the CVX Matlab toolbox [35] for solving the devised robust SOCP (RSOCP, (7)) and the benchmark robust SDP (RSDP, (9)) optimizations. The distributed method is considered only in the last sub-section. Simulations are performed on a computer with a 3.07 GHz processor and 8.0 GB of RAM. For the simulations, nodes are located in a 40 m × 40 m area. Unless specified otherwise for a simulation setting, the communication range $R_c$ in each topology is adjusted based on $|\mathcal{N}|$ for obtaining the average connectivity of $|\mathcal{K}_i| = 4$ in the network. We apply a fixed link error model with equal noise variances $\sigma_{ij}^2$ for all links $(i, j) \in \mathcal{E}$, and also a variable link error model where $\sigma_{ij}^2 = \sigma_d^2 d_{ij}^2$, and $\sigma_d^2$ is the noise variance for the unit distance, i.e., a link with a longer distance has a larger noise variance [23]. We consider the positioning mean squared error (MSE) in dBm$^2$,

$$\eta = 10 \log_{10} \left( \frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N}} \| x_i - x_i^0 \|^2 \right),$$

as the performance criterion. A smaller positioning MSE indicates a better localization performance.

A. Performance of the Robust SOCP

Figure 2 shows the RSOCP and RSDP localization results for a sample scenario from [23] with $|\mathcal{N}_g| = 10$ general sensors and $|\mathcal{N}_a| = 8$ anchors and the communication range of $R_c = 25$ m ($|\mathcal{K}_i| = 8.7$). Also, the noise variance parameter and the anchor uncertainty covariance matrix are set to $\sigma_d^2 = -20$ dBm$^2$ and $\Psi_i = -10 I_2$ dBm$^2$, respectively. As can be seen, both robust methods are able to locate anchors and general sensors with a small error. For the nodes located close to the origin $(0, 0)$, the RSOCP estimations are slightly more accurate than those for RSDP. For the nodes closer to the perimeter, the RSOCP error increases and becomes larger than the RSDP error. This trend is consistent with the fact that the nodes located closer to the center of the anchors’ convex hull are generally localized more accurately in RSOCP. Note that this property is because of the special structure in RSOCP optimization and is generally not true for RSDP. For example, the minimum RSDP localization error in Figure 2 belongs to the node located at $(-11.6, -10.8)$ m, which is far from the origin. In this scenario, the total MSE for RSOCP is
Fig. 2. Location of nodes found by the RSDP and RSOCP relaxations for the sample scenario in [23] with $|\mathcal{N}_a| = 8$ anchors, $|\mathcal{N}_s| = 10$ general sensors, $R_c = 25$ m ($\bar{R}_c = 8.7$), $\sigma_d^2 = -20$ dBm$^2$, and $\Psi_i = -10 I_2$ dBm$^2$. The blue circles and diamonds represent the actual position of general nodes and anchors, respectively; the red asterisks and plus signs show the estimated locations of general nodes and anchors by RSDP [23], respectively; and the black dots and crosses show the estimated locations of general nodes and anchors by the proposed RSOCP (7), respectively.

Table II shows the positioning MSE $\eta$, number of variables and constraints, and the CPU time spent in different robust optimizations methods for solving the network in Fig. 2. As can be seen, SOCP-based methods are faster while SDP-based methods are more accurate. Note that although robust ESDP (RESDP, [23]) has more constraints than RSDP, it is faster because the constraint sizes, i.e. number of variables involved in each constraint, are smaller.

Figure 3 shows the average value of positioning MSE as a function of $\sigma_{ij}^2$, for RSOCP, RESDP, and RSDP in 1000 uniformly-random network topologies with $|\mathcal{N}_s| = 35$ sensors and $|\mathcal{N}_a| = 15$ anchors. In order to observe the effect of anchor position uncertainties on the performance, we...
Fig. 3. The positioning MSE of the standard SOCP, RSOCP, and RSDP with and without gradient descent (GD) refinement as well as the positioning MSE of RESDP in random topologies with $|N_s| = 35$ sensors, $|N_a| = 15$ anchors, and $\Psi_i = -10 I_2$ dBm$^2$. The CRLB are also shown.

As can be seen, RSOCP outperforms the standard SOCP in terms of MSE. While RSDP performs better than RSOCP due to the tighter relaxation on the problem, we also observe that for lower values of noise (i.e. $\sigma_{ij}^2 \leq -20$ dBm$^2$), none of the convex relaxations can achieve the CRLB. This trend has also been observed in other SDP and SOCP-based localization methods, e.g., [23, Fig. 5]. The gap between the CRLB and the MSE achieved by RSDP and RSOCP can be somewhat reduced by the GD method. Not surprisingly, the RESDP performance is in between those of RSOCP and RSDP. For larger values of noise, the MSEs of all robust methods converge to the CRLB performance limit.

Figure 4 shows the average value of $\eta$ as a function of $\sigma_d^2$ with $|N_s| = 18$ sensors and $|N_a| = 12$ anchors for $\Psi_i = \kappa_i^2 I_2$ with $\kappa_i^2 = \{-10, 0, 10\}$ dBm$^2$. Also, 8 of the anchors are located at the corners similar to Figure 2. As expected, the positioning MSE is an increasing function of noise variance in all methods. The RSOCP performance is notably improved compared to the standard SOCP. Furthermore, since RSOCP is able to adjust the location of anchors, the gap between the standard and robust SOCP becomes larger by increasing anchor position uncertainties.

In order to verify the performance of the robust methods in larger scales we perform simula-
The positioning MSE, $\eta$, as a function of noise variance factor $\sigma_d^2$ in random topologies with $|N_s| = 18$ sensors, $|N_a| = 12$ anchors, and $\Psi_i = \kappa^2 I_2$, $\kappa^2 = \{-10, 0, 10\}$ dBm$^2$. 8 of the anchors are located at the corners.

The positioning MSE in a 1 km $\times$ 1 km area with $|N_s| = 105$ sensors and $|N_a| = 45$ anchors. A link failure probability of 30% is considered. The link noise variance $\sigma_{ij}^2$ changes from 20 to 34 dBm$^2$ (i.e. 10 to 50 m of noise standard deviation) and $\Psi_i = \kappa^2 I_2$, $\kappa^2 = \{34, 37.5\}$ dBm$^2$ (i.e. a radius of 50 or 75 m uncertainty around anchors). The anchor positions are randomly chosen based on a uniform distribution.

Fig. 4. The positioning MSE, $\eta$, as a function of noise variance factor $\sigma_d^2$ in random topologies with $|N_s| = 18$ sensors, $|N_a| = 12$ anchors, and $\Psi_i = \kappa^2 I_2$, $\kappa^2 = \{-10, 0, 10\}$ dBm$^2$. 8 of the anchors are located at the corners.

Fig. 5. The positioning MSE in a 1 km $\times$ 1 km area with $|N_s| = 105$ sensors and $|N_a| = 45$ anchors and show the value of $\eta$ for RSOCP and RSDP in Figure 5. In order to make the simulations closer to a real-life experiment, we also consider 30% chance of link failures in this scenario. In the presence of a link failure, no ranging data is used for the corresponding link in the optimization, i.e. nodes are not neighbours anymore. For this scenario, our RSOCP is able to localize the network in an average of 37 seconds and is more than twice faster than RSDP, which takes 84 seconds on average. The results of RSOCP MSE error provide an acceptable localization accuracy, always...
Fig. 6. Illustration of the RSOCP-RSDP combination algorithm in a sample scenario. Communication range is $R_c = 5\text{ m}$ ($|\mathcal{K}_i| = 7.4$), and the noise variance factor and uncertainty are set to $\sigma_d^2 = -20\text{ dBm}^2$, and $\Psi_i = -10I_2\text{ dBm}^2$, respectively.

within $2.5\text{ dBm}^2$ of that for RSDP. This translates to an increase of only $1.4\text{ m}$ in the standard deviation of localization error. Note that the localization algorithms would run faster in a network with a lower density compared to the results shown above, for example when the same number of sensors are deployed in a larger area.

B. Combination of RSOCP and RSDP

Next we examine the performance of the combined RSOCP and RSDP algorithm introduced in Section IV-B.

Figure 6 shows a sample network with $|\mathcal{N}_s| = 11$ sensors and $|\mathcal{N}_a| = 6$ anchors, displayed as blue circles and diamonds, respectively. First, RSOCP is executed over the entire network, resulting in the sensor and anchor estimations shown by the red squares and crosses, respectively. A threshold of $10^{-5}$ is used for determining the tight links. Note that the RSOCP sensor estimations always lie within the convex hull of estimated anchor positions (solid lines), based on Proposition 1 c). Seven of the sensors are precisely localized using RSOCP since they are tightly-connected. The total positioning MSE of RSOCP is $-4.07\text{ dBm}^2$. From the remaining four sensors which are not tight, two are located inside the convex hull. This fact shows that a node inside the convex hull is not necessarily unambiguously localizable, while the reverse is always true according to Proposition 1. The remaining four sensors, along with anchors, are re-localized using RSDP leading to the black triangle and plus signs for sensors and anchors,
Fig. 7. Average positioning MSE (solid lines) and time taken (dashed lines) for solving centralized RSDP, RSOCP and their combination for $|\mathcal{N}| = 200$ nodes as a function of the fraction of anchors, $|\mathcal{N}_a| / |\mathcal{N}|$, with $R_c = 6$ m ($|\mathcal{F}| = 9.3$), $\sigma_{ij}^2 = -35$ dBm$^2$, and $\kappa_i^2 = 0$ dBm$^2$. The anchor positions are randomly chosen based on a uniform distribution.

respectively, and a total positioning MSE of $-6.22$ dBm$^2$ for the network.

To further demonstrate the complexity advantage of mixed RSOCP-RSDP over pure RSDP, we also investigate a network with $|\mathcal{N}_s| = 105$ sensors and $|\mathcal{N}_a| = 45$ anchors (not shown here). In this scenario, RSOCP takes 33.88 seconds, and unambiguously localizes 83 sensors. The remaining 22 sensors along with anchors are passed through RSDP, resulting in the total MSE of $\eta = 0.38$ dBm$^2$ in only 6.70 seconds. In the same scenario, pure RSDP over all network takes 75% more time than the combined algorithm (71.39 seconds) and its positioning MSE is $\eta_{\text{rsdp}} = -2.46$ dBm$^2$.

The average positioning MSE (solid lines) and the total time spent (dashed lines) for RSOCP, RSDP, and the combined robust SDP-SOCP method in larger networks with $|\mathcal{N}| = 200$ sensors are shown in Figure 7. Here, the percentage of anchors $|\mathcal{N}_a| / |\mathcal{N}|$ varies from 4% to 50%. Since, unlike for the results in Figure 4, all sensors and anchors are located based on a uniform random distribution (i.e., no anchors are necessarily placed at the corners), RSOCP has a large MSE for small numbers of anchors. In this situation, the combination of RSOCP and RSDP performs significantly better than RSOCP taking twice as much time, while RSDP has the best performance with three times the computation time of RSOCP. Hence, the proposed combined method is an interesting alternative to RSOCP and RSDP, especially in sensor networks with many nodes and a relatively small number of anchors. It can be also observed from Figure 7 that as percentage of anchors increases, both positioning MSE and computation time for all three
methods decrease. The former is due to availability of more information (i.e. anchor priors) about the network, and the latter is because of the smaller $|E|$ after removing anchor-anchor links which leads to fewer variables and constraints.

C. The Distributed RSOCP Method

Now we consider larger network sizes for which it is preferable to use the distributed methods for solving the localization problem. In Figure 8, we compare the performance of the distributed RSOCP for $|N| = 100$ to 500 nodes with the three-phase iterative distributed method in [18]. We consider 10 different topologies for each value of $|N|$ and show the scatter plots for $\eta$ and computation time in Figures 8(a) and 8(b), respectively. For obtaining results in this figure, $R_c = 8$ m, 40% of the nodes are randomly selected as anchors, $\sigma_d^2 = -20$ dBm$^2$ and $\Psi_i = -10I_2$ dBm$^2$. For a fair comparison of time complexity, both algorithms are iteratively executed for only six rounds before they converge, i.e. the three-phase algorithm is run twice and our sensor-anchor refinement procedure in (17), (18) is executed three times. Also, for simplicity we have used $(0,0)$ as the initial location of all nodes in both algorithms, although intelligent initialization methods (e.g. using the centroid of neighboring anchors) may speed up the convergence. Given a sufficient number of anchors, the distributed algorithm converges to good location estimates in few iterations, as can be seen from Figure 8. Note that for both
The proposed distributed RSOCP algorithm outperforms the three-phase algorithm both in terms of \( \eta \) and computation time. The former is due to the fact that our anchor refinement step takes into account the uncertainty and is able to provide a better position estimates for anchors. The latter is because of the simpler structure of our anchor update problem compared to the proposed heuristic in [18].

### D. EM Performance

Finally, we investigate the performance of the EM algorithm when the anchor uncertainties have unknown covariance matrices. Figure 9 compares the value of \( \eta \) for the EM method, after different numbers of iteration with that for the standard and robust SOCP (with known covariance). There are \(|\mathcal{N}_s| = 35\) sensors and \(|\mathcal{N}_a| = 15\) anchors in the network with \(n_m = 5\) initial readings for each anchor, \(\sigma_{ij}^2 = -20\) dBm\(^2\), and \(\Psi_i = \kappa_i^2 I_2\), where \(\kappa_i^2\) varies from \(-15\) to \(15\) dBm\(^2\). The initial readings are obtained by generating 5 noisy versions of the actual anchor positions based on the actual \(\Psi_i\). As can be seen, the positioning MSE of EM after convergence is very close to that of RSOCP, which indicates that the EM algorithm is able to correctly
estimate the covariance matrices. On the other hand, the standard SOCP leads to a larger error, especially for larger values of uncertainty since it is unable to refine the anchor positions. For example, in Figure 9, the EM MSE after 3 to 5 iterations is close to its optimal value when \( \kappa_i^2 \geq 0 \text{ dBm}^2 \), while additional iterations are needed for smaller uncertainties.

### VII. Conclusions and Future Work

In this paper, we proposed robust and distributed algorithms for solving the localization problem based on an SOCP relaxation, which is computationally more efficient than the similar SDP methods. We analyzed the relation between the robust SOCP and robust SDP methods and provided a hybrid robust SDP-SOCP approach, which benefits from the lower complexity of robust SOCP and the better accuracy of robust SDP. The necessary and sufficient conditions for unambiguous localizability were also derived. Furthermore, an EM algorithm was proposed for estimating the locations in situations where the covariance matrices of the anchor uncertainties are unknown. Simulation results confirmed that the proposed robust, distributed, and hybrid methods based on the SOCP relaxation can be effectively used for localization in large networks with anchor position uncertainties.

### Appendix A

**Proof of Proposition 1**

We can write (7) with the relaxation (8) in the form:

\[
\nu_{\text{socp}}^* = \min_{\{x_i, q_{ij}\}} \left\| [g_{ij} | q_{ij} - d_{ij}]_{(i,j) \in E} ; [\| \Psi_i^{-1/2} (a_i - x_i) \|]_{i \in N_a} \right\|
\]

subject to \( \| x_i - x_j \| \leq q_{ij}, \ (i, j) \in E \),

(22a)

where intermediate variables \( t_{ij}, s_i, u, v \) are removed.

a) Since the Euclidean norm is strictly convex, the problem (22) must have a unique optimal with regard to all variables that appear in the norm [17]. Hence, \( x_i, \ i \in N_a \), is unique at the optimal. Note that \( x_i, \ i \in N_s \) does not appear in the norm in the objective.

b) The proof of this part is identical to [19, Proposition 5.1c], and we briefly show it here for completeness. Note that according to part a) \( q_{ij}, \ (i, j) \in E \), is unique over all solutions. For a tight link \( (i, j) \in B, \ i \in N_s \),

\[
\frac{1}{2} \| x_i - x_j \|^2 + \frac{1}{2} \| x_i' - x_j' \|^2 = \left\| \frac{x_i + x_i'}{2} - \frac{x_j + x_j'}{2} \right\|^2 + \left\| \frac{x_i - x_i'}{2} - \frac{x_j - x_j'}{2} \right\|^2,
\]

(23)
with \((x'_i, x'_j)\) being another optimal solution. Since the solution set of a convex problem is convex \([20]\), \((\frac{x_i + x'_i}{2}, \frac{x_j + x'_j}{2})\) must be another optimal solution, and thus (23) implies that 
\[
q_{ij}^2/2 + q_{ij}^2/2 = q_{ij}^2 + \|\frac{x_i - x'_i}{2} - \frac{x_j - x'_j}{2}\|^2,
\]
and thus
\[
x_i - x'_i = x_j - x'_j, \quad \forall j \in J_i,
\]
(24)

where \(J_i\) is the set of nodes which are directly connected to node \(i\) through tight links. Now, denote the set of all nodes which are connected to node \(i\) through tight links, possibly through multiple hops, by \(T_i\). According to \([19, \text{Proposition } 5.1b]\), there is at least one anchor \(x_k, k \in N\) in \(T_i\). We can then apply (24) to this anchor and its immediate tight neighbor \(x_l \in N_s\) in \(T_i\), and conclude that \(x_l = x'_l\) since \(x_k = x'_k\) holds according to part a). We can repeat the same procedure between the nodes that are proven to be invariant and their immediate neighbours to conclude that all nodes in \(T_i\) are invariant over all solutions. Now, since \(j \in T_i\), (24) implies that \(x_i\) is also invariant over all solutions of the robust SOCP.

For the proof of the reverse part, note that for a node \(x_i, i \in N_s\) with \(\|x_i - x_j\| < q_{ij}, \forall j \in K_i\), another distinct relative interior solution can be obtained as \(x'_i = x_i + \Delta x\) with \(\Delta x\) small enough such that \(\|x'_i - x_j\| < q_{ij}, \forall j \in K_i\), and all other variables remain unchanged.

c) We first show that for the analytic center solution, each sensor node is estimated within the convex hull of its neighbours, i.e. \(x_i \in \text{convexHull}\{x_j\}_{j \in K_i}, \forall i \in N_s\). Similar to \([19, \text{Proposition } 6.2]\), we argue by contradiction. Suppose that at the analytic center solution, for a sensor node \(i \in N_s, x_i\) is outside the convex hull of its neighbours. Then, let \(p_i\) be its projection on this convex hull and thus for each neighbor \(j \in K_i\) we can write,
\[
\|p_i - x_j\| < \|x_i - x_j\| \Rightarrow \log(q_{ij} - \|p_i - x_j\|) > \log(q_{ij} - \|x_i - x_j\|)
\]
\[
\Rightarrow \sum_{(i,j) \in E-B} \log(q_{ij} - \|p_i - x_j\|) > \sum_{(i,j) \in E-B} \log(q_{ij} - \|x_i - x_j\|),
\]
where the second clause follows from the uniqueness of \(q_{ij}\) over all solutions, and in the third clause, summation is taken overall non-tight links. Therefore \(x_i\) can not be the analytical center solution of node \(i\). Thus, \(x_i\) must be inside the convex hull of its neighbours. Now, since, only anchor nodes are allowed to be localized outside the convex hull of their neighbours, it follows that anchors form the convex hull of the analytic center solution.
APPENDIX B

PROOF OF PROPOSITION 2

In order to prove Proposition 2, we first construct a variation of the robust SDP optimization problem (9):

\[ v'_{sdp} = \min_{X, Y, \{M_i\}, \{\gamma_{ij}\}, \{r_{ij}\}} \left( \sum_{(i,j) \in E} g_{ij}^2 (\gamma_{ij} - 2d_{ij}r_{ij} + d_{ij}^2) + \sum_{i \in N_a} \text{tr} \left( \Psi_i^{-1}M_i \right) \right)^{\frac{1}{2}} \] (25a)

subject to

\[ \gamma_{ij} = y_{ii} + y_{jj} - y_{ij} - y_{ji}, \quad (i, j) \in \mathcal{E}, \] (25b)

\[ r_{ij}^2 \leq \gamma_{ij}, \quad (i, j) \in \mathcal{E}, \] (25c)

\[ \text{tr}(M_i) = y_{ii} - 2a_i^T x_i + a_i^T a_i, \quad i \in N_a, \] (25d)

\[ x_i = [y_{im+1} y_{im+2}]^T, \quad i \in N, \] (25e)

\[ \begin{bmatrix} y_{m+1m+1} & y_{m+1m+2} \\ y_{m+2m+1} & y_{m+2m+2} \end{bmatrix} = I_2, \] (25f)

\[ \begin{bmatrix} M_i \\ (a_i - x_i)^T \\ 1 \end{bmatrix} \succeq 0_3, \quad i \in N_a, \] (25g)

\[ Y \succeq 0_{m+2}, \] (25h)

where \( M_i \) is defined in (11). The SDP relaxation (25) is equivalent to (9) because there is a one-to-one mapping between the feasible solution set of (9) to the feasible solution set of (25). Let \( S_{sdp} = \{X, Y, \{M_i\}, \{\gamma_{ij}\}, \{r_{ij}\} \} \) be a feasible solution for robust SDP satisfying constraints (25b)-(25h). We show that the variables defined in (11) satisfy the SOCP constraints (7c)-(7e) and (8), and hence any feasible solution for robust SDP (25) (equivalently (9)) is a feasible solution for (7).

In fact, constraints (7c) and (7d) are clearly satisfied with equality by definition of \( v \) and \( t_{ij} \) in (11). To see that constraint (7e) is satisfied, first note that since SDP constraint (25g) is satisfied, \( M_i - (a_i - x_i)(a_i - x_i)^T, i \in N_a, \) is a positive semidefinite matrix. It is then clear that

\[ \text{tr} \left( \Psi_i^{-1} \left( M_i - (a_i - x_i)(a_i - x_i)^T \right) \right) \geq 0, \quad i \in N_a, \] (26)

due to the fact that the trace of the product of two positive semidefinite matrix is always non-negative. Using this inequality we obtain

\[ s_i^2 = \text{tr} \left( \Psi_i^{-1}M_i \right) \geq \text{tr} \left( \Psi_i^{-1}(a_i - x_i)(a_i - x_i)^T \right) = \| \Psi_i^{-1/2}(a_i - x_i) \|^2, \quad i \in N_a, \]
which leads to constraint (7e) by taking the square root of both sides.

For constraint (8), we follow the same line of reasoning as [21]. Specifically, since constraint (25h) is satisfied, $Y \succeq Y_m - X^TX \succeq 0_m$. Moreover, since any $2 \times 2$ principal submatrix of the positive semidefinite matrix $Z$ is also positive semidefinite, we get

$$
\begin{pmatrix}
y_{ii} - \|x_i\|^2 & y_{ij} - x_i^T x_j \\
y_{ji} - x_j^T x_i & y_{jj} - \|x_j\|^2
\end{pmatrix} \succeq 0_2, \quad (i, j) \in \mathcal{E}.
$$

(27)

Hence,

$$
\begin{pmatrix}
y_{ii} - \|x_i\|^2 & y_{ij} - x_i^T x_j \\
y_{ji} - x_j^T x_i & y_{jj} - \|x_j\|^2
\end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \geq 0, \Rightarrow

y_{ii} - \|x_i\|^2 + y_{jj} - \|x_j\|^2 - 2(y_{ij} - x_i^T x_j) \geq 0.
$$

Consequently,

$$
q_{ij}^2 = y_{ii} + y_{jj} - 2y_{ij} \geq \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = \|x_i - x_j\|^2.
$$

Therefore, constraint (8) is also satisfied.

We have shown that all constraints of the robust SOCP are satisfied and thus any feasible solution of the robust SDP is also a feasible solution of the robust SOCP. Also, the optimum value of the robust SOCP in (7) is smaller than or equal to that for the robust SDP in (25) because at the optimal point of SDP, which is a feasible point for SOCP, we have

$$
v'_{\text{sdp}} = \sqrt{v_{\text{sdp}}} + v_0 = \left( \sum_{(i, j) \in \mathcal{E}} g_{ij}^2 (\gamma_{ij} - 2d_{ij}r_{ij} + d_{ij}^2) + \sum_{i \in \mathcal{N}_a} \text{tr} \left( \Psi_i^{-1} M_i \right) \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{(i, j) \in \mathcal{E}} g_{ij}^2 (\gamma_{ij} - 2d_{ij}\sqrt{\gamma_{ij}} + d_{ij}^2) + \sum_{i \in \mathcal{N}_a} s_i^2 \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{(i, j) \in \mathcal{E}} \left( q_{ij} - d_{ij} \right)^2 + \sum_{i \in \mathcal{N}_a} s_i^2 \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{(i, j) \in \mathcal{E}} \tilde{t}_{ij}^2 + \sum_{i \in \mathcal{N}_a} s_i^2 \right)^{\frac{1}{2}} = v_{\text{socp}},
$$

where the third equality holds because at the optimum solution of the robust SDP, $r_{ij}^2 = \gamma_{ij}$, i.e. constraint (25e) is satisfied with equality [25]. The proof is complete by noting that, by definition, any non-optimal solution of SDP has a value of objective which is larger than $v'_{\text{sdp}}$. ■
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