Fixed-Complexity Piecewise Ellipsoidal Representation of the Continual Reachability Set Based on Ellipsoidal Techniques

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Motivation: Control of Anesthesia

- Control of depth of anesthesia
  - [Simanski, et al. 07; Ionescu, et al. 08; Syafiie, et al. 09; Dumont, et al. 09; Oliveira, et al. 09; Mendonca, et al. 09]

- Goal: closed-loop drug delivery system
- For FDA/Health Canada: guarantees of safety and performance
- Patient-oriented control design
Motivation: Other Application Domains

- Fleet of environmental monitoring motes with limited power
- Must be dispersed using bounded input authority
- Alert depletion of battery at least $t_a$ time units in advance
- Objectives:
  - Return to the base upon low-battery alert
  - Spend maximum possible time outside
  - Roam over as large of an area outside of the base as possible
Introduction

\[ \Delta x = f(x, u), \quad x(0) = x_0 \]
\[ u(t) \in U \quad \text{(input constraint)} \]
\[ K \subseteq X \quad \text{(target set/state constraint)} \]

- Reachability analysis
  - [Tomlin, et al. 03; Aubin, et al. 11; Kurzhanski and Varaiya 00; Lygeros 04; Blanchini and Miani 08; ...]
- Typically used to guarantee safety
  - [Lygeros, et al. 98; Mitchell, et al. 05; Bayen, et al. 07; ...]
- Safety constraints may be temporarily relaxed in favor of improved performance
  - **Continual reachability set** to explore that option [CDC-2011]
Outline

Continual Reach Set (Overview)

Implications and Numerical Approximation

Fixed-Complexity Piecewise Ellipsoidal Representation

Application: Control of Anesthesia
The Continual Reach Set

\[ \text{Reach}^\gamma_{[0,\tau]}(\mathcal{K}) := \{ x_0 \in \mathcal{X} \mid \forall t, \exists u(\cdot), x_{x_0}^{u(\cdot)}(t) \in \mathcal{K} \} \]

- Set of states that can reach \( \mathcal{K} \) at any time within the finite horizon
- For any desired time there exists at least one input policy that can steer the system to the target
- Additional flexibility to a supervisory controller: a trade-off between the desired time-to-reach the target and the input effort
The Continual Reach Set (cont’d)

Target/Constraint Set

\[ \mathcal{K} \rightarrow \text{Target/Constraint Set} \]
The Continual Reach Set (cont’d)

Viability Kernel

\[ \mathcal{K} \]

\[ Viab_{[0,\tau]}(\mathcal{K}) \]
Continual Reach Set

\[ \mathcal{K} \]

\[ \text{Reach}^{\gamma}_{[0,\tau]}(\mathcal{K}) \]

\[ \text{Viab}_{[0,\tau]}(\mathcal{K}) \]
Implications

- What it means for performance
- More flexibility since viability control laws are a subset
- Performance + safety; a mixed scheme:

\[ \text{Reach}^{\gamma}_{[0,\tau]}(\text{Viab}[0,\tau](\mathcal{K})) = \text{Viab}_{[0,\tau]}(\mathcal{K}) \]

- The original viability control laws are still a subset
Implications

- What it means for performance
- More flexibility since viability control laws are a subset
- Performance + safety; a mixed scheme:

\[
\text{Reach}^{\gamma}_{[0,\tau]}(\text{Viab}_{[0,\tau]}(\mathcal{K})) = \text{Viab}_{[0,\tau]}(\mathcal{K})
\]

- The original viability control laws are still a subset
Numerical Approximation

• Approximation via ellipsoidal techniques for DT linear systems
  [Kurzhanski and Varaiya 00; CDC-2011]

\[
Reach_{[0,\tau]}(\mathcal{K}) \supseteq Reach_{[0,\tau]}(\mathcal{K}_{\downarrow \varepsilon}) \\
\supseteq \bigcap_{t \in [0,\tau]} \left( \bigcup_{\ell_{\tau} \in \mathcal{J}} \mathcal{E}(x^*(t), X_{\ell_{\tau}}^{-}(t)) \right)
\]
• Simple, closed-form representation is desired
• Interchanging order of quantifiers

\[
\text{Reach}^{\gamma}_{[0, \tau]}(K_{\downarrow \epsilon}) \supseteq \bigcap_{t \in [0, \tau]} \left( \bigcup_{\ell_{\tau \in J}} E(x^*(t), X_{\ell}(t)) \right)
\]
\[
\supseteq \bigcup_{\ell_{\tau \in J}} \left( \bigcap_{t \in [0, \tau]} E(x^*(t), X_{\ell}(t)) \right)
\]

Ellipsoid via SDP

• Pick at most \( k \leq |J| \) ellipsoids s.t. maximal volume and connected
  ▶ Maximum \( k \)-Connected Ellipsoidal Representation (kCER)
Fixed-Complexity Piecewise Ellipsoidal Representation

- kCER $\leftrightarrow$ binary integer program

$$\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{\left|\mathcal{J}\right|} (c_i - d_i) z_i \\
\text{s. t.} & \quad c_i = \text{vol}(\mathcal{E}_i), \quad d_i = \sum_{j=1}^{\left|\mathcal{J}\right|} \frac{\text{vol}(\mathcal{E}_i \cap \mathcal{E}_j)}{\text{vol}(\mathcal{E}_i)} \\
& \quad \sum_{i=1}^{\left|\mathcal{J}\right|} z_i \leq k \\
& \quad \sum_{i=1}^{\left|\mathcal{J}\right|} \sum_{j=1}^{\left|\mathcal{J}\right|} z_i \left(\gamma_{ij}^{(k-1)} - 1\right) z_j = 0 \quad \text{graph-theoretic approach for connectedness}
\end{align*}$$
Application: Control of Anesthesia

- Discrete-time Laguerre model; patient’s response to rocuronium
- Target set: therapeutic bounds on output (pseudo-occupancy level), i.e. desired clinical effect
- Input constraint: actuator bounds (hard bounds on drug infusion rate)
- Reformulate by projecting the output bounds onto the state space (7D) while making the control action regulatory
A projection of the continual reach set and its 3-connected piecewise ellipsoidal representation computed using Ellipsoidal Toolbox.
(patient #80. 60 min surgery.)
Application: Control of Anesthesia (cont’d)

- Implemented as additional constraints in MPC
- A guarantee of performance; desired clinical effect can be reached at arbitrary times
- Minimize total administered drug, or achieve a desired depth of anesthesia arbitrarily fast
- Optimal infusion rate to keep within the target clinical effect may not be physiologically ideal (discontinuous/bang-bang)
- May choose to temporarily relax the state constraint in exchange for a better-suited (less aggressive, mildly varying) infusion rate
- Physiologically more optimized to meet the operating conditions and patient’s ability to handle drug (patient-oriented design)
Conclusions and Future Work

• Continual reach set to guarantee performance
• Simple, closed-form approximation of fixed complexity
• The Maximum $k$-Connected Ellipsoidal Representation as variation of Maximum Coverage Problem
• Ensured connectedness via graph theory
• Application to anesthesia: Facilitates physiologically relevant control

• Quantifying approximation error
• Synthesizing continual reachability control laws
• Accounting for model uncertainty
• Implementation of a safety- + performance-based controller
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Some Properties

• For every \( x_0 \in \text{Reach}^{\gamma}_{[0, \tau]}(\mathcal{K}) \) and some \( t \in [0, \tau] \),

\[
d(x_{x_0}^{u(\cdot)}(\hat{t}), \mathcal{K}) \leq d_{H_1}(\text{Reach}^\#_{t-\hat{t}}(\mathcal{K}), \mathcal{K}) \quad \forall \hat{t} \in [0, t]
\]

for any \( u(\cdot) \in \mathcal{U}_{[0, t]} \) s.t. \( x_{x_0}^{u(\cdot)}(t) \in \mathcal{K} \)

• States inside maximal reach tube but outside continual reach set can only reach target at specific times
• Fleet of environmental monitoring motes with limited power source
• Must be dispersed using bounded input authority
• Alert depletion of battery at least $t_a$ time units in advance
• Objectives:
  ▶ Return to the base upon low-battery alert
  ▶ Spend maximum possible time outside
  ▶ Roam over as large of an area outside of the base as possible
• Solution: $\text{Reach}^\gamma_{[t_a,\tau]}(\mathcal{K})$