Fixed-Complexity Piecewise Ellipsoidal Representation of the Continual Reachability Set Based on Ellipsoidal Techniques

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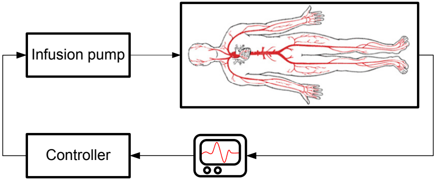
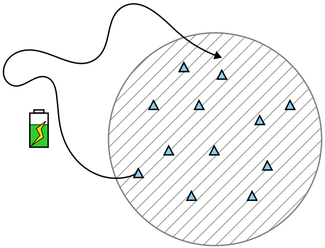
Motivation: Control of Anesthesia

- Control of depth of anesthesia
  - [Simanski, et al. 07; Ionescu, et al. 08; Syafiie, et al. 09; Dumont, et al. 09; Oliveira, et al. 09; Mendonca, et al. 09]
- Goal: closed-loop drug delivery system
- For FDA/Health Canada: guarantees of safety and performance
- Patient-oriented control design



Motivation: Other Application Domains

- Fleet of environmental monitoring motes with limited power
- Must be dispersed using bounded input authority
- Alert depletion of battery **at least** t_a time units in advance
- Objectives:
  - Return to the base upon low-battery alert
  - Spend maximum possible time outside
  - Roam over as large of an area outside of the base as possible



Introduction

$$\begin{cases} \Delta x = f(x,u), & x(0) = x_0 \\ u(t) \in \mathcal{U} & \text{(input constraint)} \\ \mathcal{K} \subseteq \mathcal{X} & \text{(target set/state constraint)} \end{cases}$$

- Reachability analysis
  - [Tomlin, et al. 03; Aubin, et al. 11; Kurzhanski and Varaiya 00; Lygeros 04; Blanchini and Miani 08; ...]
- Typically used to guarantee safety
  - [Lygeros, et al. 98; Mitchell, et al. 05; Bayen, et al. 07; ...]
- Safety constraints may be temporarily relaxed in favor of improved performance
  - **Continual reachability set** to explore that option [CDC-2011]

Outline

Continual Reach Set (Overview)

Implications and Numerical Approximation

Fixed-Complexity Piecewise Ellipsoidal Representation

Application: Control of Anesthesia

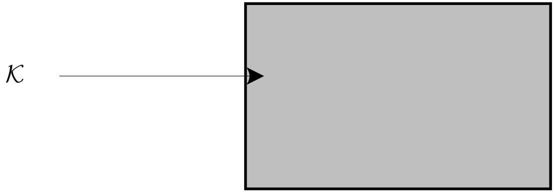
The Continual Reach Set

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) := \left\{x_0 \in \mathcal{X} \mid \forall t,\ \exists u(\cdot),\ x^{u(\cdot)}_{x_0}(t) \in \mathcal{K}\right\}$$

- Set of states that can reach $\mathcal{K}$ at any time within the finite horizon
- For any desired time there exists at least one input policy that can steer the system to the target
- Additional flexibility to a supervisory controller: a trade-off between the desired time-to-reach the target and the input effort

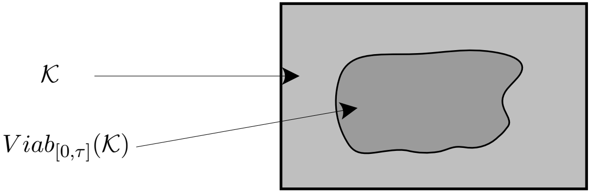
The Continual Reach Set (cont'd)

Target/Constraint Set

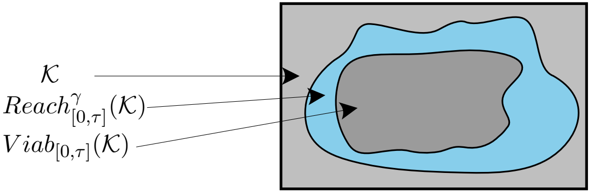


The Continual Reach Set (cont'd)

Viability Kernel



Continual Reach Set

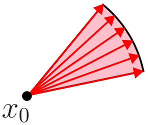


Implications

- What it means for performance
- More flexibility since viability control laws are a subset
- Performance + safety; a mixed scheme:

$$Reach^{\gamma}_{[0,\tau]}(Viab_{[0,\tau]}(\mathcal{K})) = Viab_{[0,\tau]}(\mathcal{K})$$

- The original viability control laws are **still a subset**



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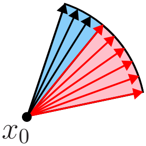
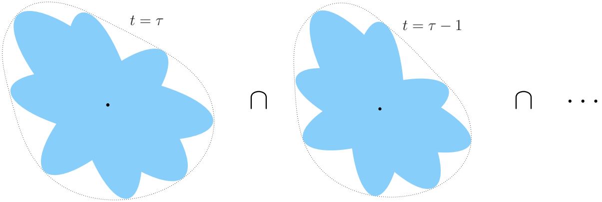


Numerical Approximation

- Approximation via ellipsoidal techniques for DT linear systems
  [Kurzhanski and Varaiya 00; CDC-2011]

$$
\begin{aligned}
Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) &\supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \\
&\supseteq \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau \in \mathcal{J}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right)
\end{aligned}
$$



Fixed-Complexity Piecewise Ellipsoidal Representation

- Simple, closed-form representation is desired
- Interchanging order of quantifiers

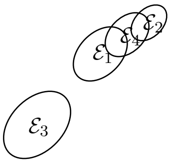
$$
\begin{aligned}
Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) &\supseteq \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau \in \mathcal{J}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right) \\
&\supseteq \bigcup_{\ell_\tau \in \mathcal{J}} \underbrace{\left(\bigcap_{t\in[0,\tau]} \mathcal{E}(x^*(t), X^-_\ell(t)) \right)}_{\text{Ellipsoid via SDP}}
\end{aligned}
$$

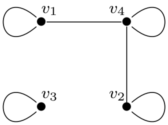
- Pick at most $k \leq |\mathcal{J}|$ ellipsoids s.t. **maximal volume** and **connected**
  - Maximum k-Connected Ellipsoidal Representation (kCER)

Fixed-Complexity Piecewise Ellipsoidal Representation

- kCER $\longleftrightarrow$ binary integer program

$$\begin{aligned}
\underset{z_1,\ldots,z_{|\mathcal{J}|}\in\{0,1\}}{\text{maximize}} \quad & \textstyle\sum_{i=1}^{|\mathcal{J}|} (c_i - d_i)\, z_i \\
\text{s. t.} \quad & c_i = \text{vol}(\mathcal{E}_i),\ d_i = \textstyle\sum_{j=1}^{|\mathcal{J}|} \frac{\text{vol}(\mathcal{E}_i \cap \mathcal{E}_j)}{\text{vol}(\mathcal{E}_i)} \\
& \textstyle\sum_{i=1}^{|\mathcal{J}|} z_i \leq k \\
& \underbrace{\textstyle\sum_{i=1}^{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} z_i(\gamma_{ij}^{(k-1)} - 1) z_j = 0}_{\text{graph-theoretic approach for connectedness}}
\end{aligned}$$

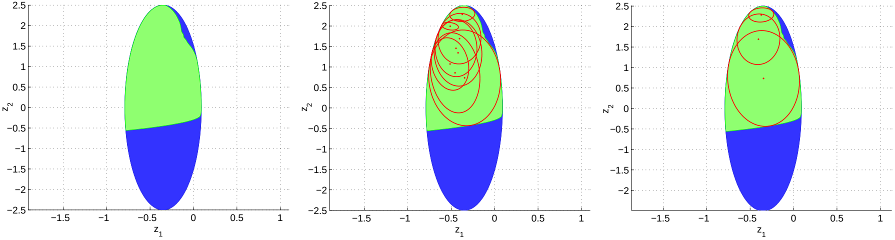




Application: Control of Anesthesia

- Discrete-time Laguerre model; patient's response to rocuronium
- Target set: therapeutic bounds on output (pseudo-occupancy level), i.e. desired clinical effect
- Input constraint: actuator bounds (hard bounds on drug infusion rate)
- Reformulate by projecting the output bounds onto the state space (7D) while making the control action regulatory

Application: Control of Anesthesia (cont'd)



A projection of the continual reach set and its 3-connected piecewise ellipsoidal representation computed using Ellipsoidal Toolbox. (patient $\#80$. 60 min surgery.)

Application: Control of Anesthesia (cont'd)

- Implemented as additional constraints in MPC
- A guarantee of performance; desired clinical effect can be reached at arbitrary times
- Minimize total administered drug, or achieve a desired depth of anesthesia arbitrarily fast
- Optimal infusion rate to keep within the target clinical effect may not be physiologically ideal (discontinuous/bang-bang)
- May choose to temporarily relax the state constraint in exchange for a better-suited (less aggressive, mildly varying) infusion rate
- Physiologically more optimized to meet the operating conditions and patient's ability to handle drug (patient-oriented design)

Conclusions and Future Work

- Continual reach set to guarantee performance
- Simple, closed-form approximation of fixed complexity
- The Maximum k-Connected Ellipsoidal Representation as variation of Maximum Coverage Problem
- Ensured connectedness via graph theory
- Application to anesthesia: Facilitates physiologically relevant control

- Quantifying approximation error
- Synthesizing continual reachability control laws
- Accounting for model uncertainty
- Implementation of a safety- + performance-based controller

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NSERC CRSNG

CIHR IRSC
Canadian Institutes of Health Research
Instituts de recherche en santé du Canada

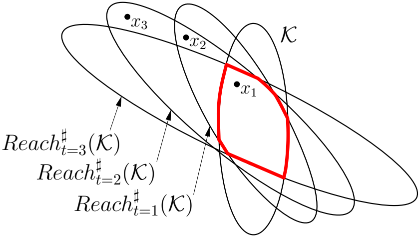
ICICS CONNECTING KNOWLEDGE

Some Properties

- For every $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ and some $t \in [0,\tau]$,

$$d(x^{u(\cdot)}_{x_0}(\hat{t}), \mathcal{K}) \leq d_{H_1}\left(Reach^{\sharp}_{t-\hat{t}}(\mathcal{K}), \mathcal{K}\right) \quad \forall \hat{t} \in [0,t]$$

  for any $u(\cdot) \in \mathscr{U}_{[0,t]}$ s.t. $x^{u(\cdot)}_{x_0}(t) \in \mathcal{K}$
- States inside maximal reach tube but outside continual reach set can only reach target at specific times



Other Application Domains (revisited)

- Fleet of environmental monitoring motes with limited power source
- Must be dispersed using bounded input authority
- Alert depletion of battery **at least** t_a time units in advance
- Objectives:
  - Return to the base upon low-battery alert
  - Spend maximum possible time outside
  - Roam over as large of an area outside of the base as possible
- Solution: $Reach^{\gamma}_{[t_a,\tau]}(\mathcal{K})$