NUMERICAL INTEGRATION
ACCURACY AND STABILITY
NUMERICAL INTEGRATION

- Time discretization. Integration rules.
- Accuracy of Integration rules.
- Distortion of Network Parameters.
- Numerical Stability
- Critical Damping Adjustment (CDA)
1. **TIME DISCRETIZATION**

- Closed-form analytical solutions of large systems with frequency dependent parameters, switches, and nonlinearities are, in general, very difficult, or at all not possible.

- Best technique in systems with switches and nonlinearities is to discretize time and solve the system equations on successive time steps, i.e., at $t = 0, \Delta t, 2\Delta t,$ ...

- There are limitations when the system is solved this way:
  
  - Size of $\Delta t$
  
  - Discretization Rule
    (e.g., trapezoidal, backward Euler, etc.)
2. **FROM DIFFERENTIAL TO DIFFERENCE EQUATIONS**

The behaviour of a simple inductance (or capacitance) allows us to define discrete-time differentiation or integration.

**Discrete-Time Differentiation**

\[
U = L \frac{d}{dt} i(t)
\]

\[
\frac{i(t) - i(t-\Delta t)}{\Delta t} = \frac{1}{L} U(t)
\]

**Attempts**

a) \( \frac{i(t) - i(t-\Delta t)}{\Delta t} = \frac{1}{L} U(t) \) \( t_1 = t \) (Backward Euler)

b) \( \frac{i(t) - i(t-\Delta t)}{\Delta t} = \frac{1}{L} U(t-\frac{\Delta t}{2}) \) \( t_1 = \text{middle} \)

c) \( \frac{i(t) - i(t-2\Delta t)}{2\Delta t} = \frac{1}{L} U(t-\Delta t) \) (Mid-point rule)

\( \frac{i(t) - i(t-\Delta t)}{\Delta t} = \frac{1}{L} \frac{U(t) + U(t-\Delta t)}{2} \) Averages not the t’s but the function at end points (Trapezoidal)

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3. DISCRETE-TIME INTEGRATION

\[ U(t) \]

\[ \text{Simpson} \quad \text{Backward Euler} \quad \text{Trapezoidal} \quad \text{Forward Euler} \]

\[ i(t) \quad L \quad 0 \quad \frac{U(t)}{} \]

\[ \int_{t-\Delta t}^{t} U dt = L \int_{t-\Delta t}^{t} di \]

\[ \Delta t = L i(t) - L i(t-\Delta t) \]

a) Trapezoidal: \[ \text{area} = \frac{U(t) + U(t-\Delta t)}{2} \Delta t = L i(t) - L i(t-\Delta t) \]

b) Backward Euler: \[ \text{area} = U(t) \cdot \Delta t = L i(t) - L i(t-\Delta t) \]

c) Forward Euler: \[ \text{area} = U(t-\Delta t) \cdot \Delta t = L i(t) - L i(t-\Delta t) \]

d) Simpson (parabola): \[ \text{area} = \left[ \frac{1}{3} U(t) + \frac{4}{3} U(t-\Delta t) + \frac{1}{3} U(t-2\Delta t) \right] \Delta t = L i(t) - L i(t-2\Delta t) \]

e) Gear 2nd Order (differentiation): \[ \frac{3}{2} \left[ i(t) - \frac{4}{3} i(t-\Delta t) + \frac{1}{3} i(t-2\Delta t) \right] = \frac{\Delta t}{L} U(t) \]

* UNSTABLE
Accuracy of discretization rules

Analysis Techniques

- Usually in terms of truncation error in time domain solution.
- In circuit analysis frequency response is very important.
- We can more accurately assess performance of discretization rules in the frequency domain.
1. FREQUENCY RESPONSE OF A LINEAR SYSTEM

Take input that has only one frequency:

\[ X(t) = e^{j\omega t} = 1 \]

(vector of magnitude one, rotating at velocity \( \omega \))

If the system is linear and has no delays, the response will have the form

\[ Y(t) = X(t) \cdot H(\omega) = H(\omega) e^{j\omega t} \]

where \( H(\omega) \) is the magnitude and phase "gain" and equals the transfer function \( H(s) \) evaluated at \( s = j\omega \),

\[ H(\omega) = H(s) \bigg|_{s = j\omega} \]
2. **FREQUENCY RESPONSE OF AN INTEGRATOR**

The response of an inductance when \( V \) is input and \( i \) is output gives us the response of an integrator.

**Continuous-Time Response**

**Input:** \( V(t) = e^{j\omega t} \)

**Output:** \( i(t) = Y(\omega) e^{j\omega t} \) to be found

With \( V = L \frac{di}{dt} \),

\[ e^{j\omega t} = L Y \omega j e^{j\omega t} \Rightarrow Y(\omega) = \frac{1}{j\omega L} \] or \( Y(s) = \frac{1}{sL} \) for \( s = j\omega \)

The pure integrator is obtained with \( L = 1 \).

\[ Y(\omega) = \frac{1}{j\omega L} \]

*Admittance of an \( L \) in the continuous-time frequency domain*

\[ F(\omega) = \frac{1}{j\omega L} \]

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Response of Integrator (Cont.)

Discrete-Time Response

With Trapezoidal,

\[ i(t) - i(t - \Delta t) = \frac{\Delta t}{2L} V(t) + \frac{\Delta t}{2L} V(t - \Delta t) \]

**input:** \( V(t) = e^{j\omega t} \)

**output:** \( i(t) = Y_e e^{j\omega t} \) to be found

Substituting,

\[ Y_e e^{j\omega t} - Y_e e^{j\omega(t - \Delta t)} = \frac{\Delta t}{2L} e^{j\omega t} + \frac{\Delta t}{2L} e^{j\omega(t - \Delta t)} \]

Factoring out \( e^{j\omega t} \),

\[ Y_e(\omega) = \frac{\Delta t}{2L} \frac{e^{j\omega \Delta t} + 1}{e^{j\omega \Delta t} - 1} \]

*Admittance of an \( L \) in the discrete-time frequency domain.*

\[ \begin{align*}
\text{diagram with circuit and equations}
\end{align*} \]
Response of Integrator (Cont. 2)

Accuracy of Integrator

The accuracy of the discrete-time integrator can be expressed by the ratio

\[ \frac{H_e(\omega)}{H(\omega)} = \left. \frac{Y_e(\omega)}{Y(\omega)} \right| \omega L = 1 \]

\[ = \left( \frac{\Delta t}{2} \right) \frac{e^{j\omega \Delta t} + 1}{1 - j\omega \Delta t} \]

\[ \frac{H_e(\omega)}{H(\omega)} = j \left( \frac{\Delta t}{2} \right) \frac{1 - j\omega \Delta t}{1 - j\omega \Delta t} \]

The maximum frequency that may be present in the signals is given by the Sampling Theorem,

\[ f_{Ny} = \frac{1}{2\Delta t} = 0.5 \text{ pu} \]

The response beyond this point is unimportant because there are no frequencies beyond this point in the circuit.

\[ \text{Tr} \]

\[ \text{Ideal} \]

\[ f_{\text{Base}} = \frac{1}{\Delta t} \]

\[ f(\text{pu}) = f(H_g) \cdot \Delta t \]

\[ \text{Since } f(H_g) = \frac{f(\text{pu})}{\Delta t} \]

To move up the frequency limit, decrease the \( \Delta t \)
3. EXAMPLE OF DISCRETIZATION EFFECTS

a) Continuous-Time Solution

\[ \bar{V} = 1000 \angle 20^\circ \text{ V (RMS)} \]
\[ \bar{Y} = \frac{1}{j\omega L} = \frac{1}{2\pi \times 4000 \times 20 \times 10^{-3}} \]
\[ \bar{Y} = 0.0020 \angle -90^\circ \text{ S} \]
\[ \bar{I} = \bar{V} \bar{Y} = 2 \angle -70^\circ \text{ A (RMS)} \]

and with the \( \sqrt{2} \) factor to convert from RMS to peak,

\[ i(t) = 2.83 \cos (\omega t - 70^\circ) \text{ A} \]

Exact Solution.

> What we would see on an oscilloscope.
Example of Discretization Effects (Cont.)

b) Discrete-Time Solution with Trapezoidal

i) Solution with $\Delta t = 0.1 \text{ ms}$

\[
\bar{V} = 1000 \left[ 20^\circ \text{ V} \right] \text{ given.}
\]

\[
\bar{V}_e = \frac{\Delta t}{2L} \frac{e^{j\omega t} + 1}{e^{j\omega t} - 1} = \frac{10^{-4}}{4 \times 10^{-3}} \frac{1}{1 - 1 - 40^\circ}
\]

\[
= 0.0025 \frac{1}{144^\circ - 1}
\]

\[
\bar{V}_e = 0.00081 \left[ -90^\circ \text{ S} \right] \text{ Way off correct}
\]

\[
\bar{I} = \bar{V}_e \bar{V} = 0.81 \left[ -70^\circ \right]
\]

\[
l(t) = 1.15 \cos (\omega t - 70^\circ) \text{ A}
\]

For the chosen $\Delta t$,

\[
\left( 59\% \text{ error} \right) \text{ correct}
\]

\[
\frac{f_{Ny}}{2\Delta t} = \frac{1}{2 \times 10^{-4}} = 5000 \text{ Hz}
\]

Source $= 4,000 \text{ Hz}$ which is too close to $f_{Ny}$

(It should be $\frac{1}{2} \text{ to } \frac{1}{4}$)

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ii) Solution with $\Delta t = 0.01 \text{ ms}$

\[
\overline{v} = 1000 \frac{\text{m}}{\text{s}} \quad \text{given}
\]

\[
\overline{v} = \frac{\Delta t}{2\pi} \frac{e^{j\omega \Delta t} + 1}{e^{j\omega t} - 1} = \frac{10^{-5}}{40 \times 10^{-3}} \frac{1 + 1}{1 - 14.4^\circ - 1}
\]

\[
\overline{y_e} = 0.0020 \underbrace{| -90^\circ |}_{\text{correct! correct}} \quad S
\]

\[
\overline{V} = \overline{y_e} \overline{V} = 2 | -70^\circ | \quad A
\]

\[
\overline{i}(t) = 2.83 \cos(\omega t - 70^\circ) \quad A
\]

\[
\text{We would now get the correct result with SPICE or the EMTP!}
\]

Notice that now,

\[
f_{Ny} = \frac{1}{2\Delta t} = \frac{1}{2 \times 10^{-5}} = 50,000 \text{ Hz}
\]

and the source frequency of 4,000 Hz is

\[
\frac{1000}{50,000} < \frac{1}{10} \text{ of } f_{Ny}
\]
4. DISTORTION OF NETWORK PARAMETERS

Distortion of L and C by Trapezoidal

With trapezoidal

\[ Z_e = \frac{1}{Y_e} = \frac{2L}{\Delta t} \frac{e^{j\omega \Delta t} - 1}{e^{j\omega \Delta t} + 1} \]
\[ = \frac{2L}{\Delta t} \frac{e^{j\omega \Delta t/2} - e^{-j\omega \Delta t/2}}{e^{j\omega \Delta t/2} + e^{-j\omega \Delta t/2}} \]
\[ Z_e = \left(\frac{2L}{\Delta t}\right) j \tan \left(\frac{\omega \Delta t}{2}\right) \]

Defining an equivalent L such that
\[ Z_e = j\omega L_e \]  
(by analogy with continuous time)

\[ L_e(\omega) = L \frac{\tan \left(\frac{\omega \Delta t/2}{2}\right)}{\left(\frac{\omega \Delta t}{2}\right)} \]

Continuous Time
\[ L \quad j\omega L \]

Discrete-Time with Trapezoidal
\[ R = \frac{2L}{\Delta t} \quad \frac{\omega L_e}{\Delta t} \quad \text{Frequency Domain} \]

Equivalent
\[ L_e(\omega) \quad \frac{\omega L_e}{\Delta t} \]

For \( \omega \Delta t \to 0 \), \( L_e \to L \)
For \( \frac{\omega \Delta t}{2} = \frac{\pi}{2} \), \( L_e \to \infty \)
\[ \omega \Delta t = \pi, \quad 2\pi f \Delta t = \pi \]
\[ f = \frac{1}{2\Delta t} = f_{Ny} \]
\[ \frac{L_e}{L} \to 0 \quad \frac{L_e}{L} \to \infty \]
\[ f \to 0 \quad f \to \frac{1}{2\Delta t} = f_{Ny} \]

Which is what the continuous-time L is when \( f \to \infty \). There is a frequency warping effect, but

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Distortion of L and C by Trapezoidal (Cont.)

If the analysis is repeated for a capacitance \( C \), the same distortion factor is found:

\[
L_e = k(\omega) L \\
C_e = k(\omega) C
\]

\[
K(\omega) = \frac{\tan \left( \frac{\omega \Delta t}{2} \right)}{\left( \frac{\omega \Delta t}{2} \right)}
\]

\[
X_e = \omega L_e = \omega k(\omega) L = \omega e L \\
B_e = \omega C_e = \omega k(\omega) C = \omega e C
\]

The distortion on the \( L \) and \( C \) can also be viewed as a "frequency warping" effect, where \( \omega e = k(\omega) \omega \) and infinity is located at \( f = \frac{1}{2\Delta t} \).
5. DISTORTION OF L AND C BY BACKWARD EULER RULE

With the backward Euler rule, for an inductance \( L \)

\[
i(t) - i(t - \Delta t) = \frac{\Delta t}{L} \, V(t)
\]

input: \( V(t) = e^{j\omega t} \)

output: \( i(t) = Ye \, e^{j\omega t} \)

Substituting,

\[
Ye \, e^{j\omega t} - Ye \, e^{j\omega (t - \Delta t)} = \frac{\Delta t}{L} \, e^{j\omega t}
\]

\[
Ye(\omega) = \frac{\Delta t}{L} \, \frac{e^{j\omega \Delta t}}{e^{j\omega \Delta t} - 1}
\]

or,

\[
Ye(\omega) = \frac{\Delta t}{L} \, \frac{e^{j\omega \Delta t/2}}{e^{j\omega \Delta t/2} - e^{-j\omega \Delta t/2}}
\]

\[
Ye(\omega) = \frac{\Delta t}{L} \, \frac{\cos \frac{\omega \Delta t}{2} + j \sin \frac{\omega \Delta t}{2}}{2j \sin \frac{\omega \Delta t}{2}}
\]

\[
Ye(\omega) = \left( \frac{\Delta t}{2L} \right) \left( \frac{\Delta t}{2L} \right) \frac{1}{j \tan \left( \frac{\omega \Delta t}{2} \right)}
\]

From the form of \( Ye(\omega) \),

\[
Ye = \frac{1}{Re} + \frac{1}{j\omega Le}
\]

\[
Re = \frac{2L}{\Delta t} \quad \text{Resistance}
\]

\[
Le = L \left( \frac{\tan \left( \frac{\omega \Delta t}{2} \right)}{\frac{\omega \Delta t}{2}} \right) \quad \text{Same as for \text{Trapezoidal dist.}}
\]

\[
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\]
6. TRAPEZOIDAL VS. BACKWARD EULER

<table>
<thead>
<tr>
<th>Circuit Element</th>
<th>Trapezoidal</th>
<th>Backward Euler</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$L_{e}$</td>
<td>$L_{e}$</td>
</tr>
<tr>
<td>$C$</td>
<td>$C_{e}$</td>
<td>$C_{e}$ &lt;sup&gt;Re = $\frac{\alpha}{2C}$&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

$$L_{e} = L \frac{\tan\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega t}{2}\right)}$$

$$C_{e} = C \frac{\tan\left(\frac{\omega t}{2}\right)}{\left(\frac{\omega t}{2}\right)}$$

In addition to the distortion in the value of the parameter $L$ or $C$, backward Euler adds a fictitious resistance to the circuit. This resistance produces additional losses but no relative phase distortion. All frequencies remain synchronized.
7. COMPARISON OF FREQUENCY RESPONSES

- All rules give good magnitude response for frequencies up to 0.05 to 0.1 in p.u.
  \[ \Rightarrow \frac{1}{10} \text{th to } \frac{1}{5} \text{th of } f_{\text{Ny}} \]

- Simpson has very good response but is unstable.

- Trapezoidal does not introduce phase distortion but can have problems at discontinuities.

- Backward Euler has no problems at discontinuities but presents strong phase distortion.

- Gear is a compromise between trapezoidal and backward Euler.

- CDA combines best of trap with best of B.E.
STABILITY OF DISCRETIZATION RULES

- Difference Equations
- Z-Domain Transfer Functions
- Critical Damping Adjustment CDA.
1. DIFFERENCE EQUATIONS

Solution of a Differential Equation

\[ R \frac{di}{dt} + L \frac{d^2i}{dt^2} = U \]
\[ \frac{d^2i}{dt^2} + \frac{R}{L} i = \frac{1}{L} U \]

Total Solution = Steady-state + Transient

a) Transient

From "homogeneous solution" (no forcing function)

\[ \frac{d^2i}{dt^2} + \frac{R}{L} i = 0 \]

Solution has the form \( AE^{rt} \)

\[ r^2 + \frac{R}{L} = 0 \]
\[ r = -\frac{R}{L} \]

\[ i_h(t) = C e^{-\frac{R}{L}t} \]

Assume \( U(t) \) is input

Final state transition from initial state to final steady-state

\[ \frac{d}{dt} \]

\[ \frac{E}{R} \]

Initial state \( \rightarrow \) Transient

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Solution of a Differential Equation (Cont.)

b) Steady State
A "particular" solution, according to the input or "forcing function" is of the form \( l_{ss} = \frac{E}{R} \). The steady-state solution is:

\[
\frac{R}{L} B = \frac{1}{L} E \Rightarrow B = \frac{E}{R}
\]

(Could be written directly from the circuit).

\( l_{ss}(t) = \frac{E}{R} \)

(c) Complete Solution

\[
i(t) = i(t) + l_{ss}(t) = C e^{-\frac{R}{L} t} + \frac{E}{R}
\]

\( \text{Pole is negative} \)
\( \text{grows to zero as } t \to \infty \Rightarrow \text{STABLE}! \)

For a system to be stable \(-\text{transient} \to \text{dies out} \)

The form of the transient solution is independent of the initial and final states. The arbitrary constant does the job of matching the states.

In the example,

\[
t = 0, \quad i = 0 \Rightarrow C + \frac{E}{R} = 0 \Rightarrow C = -\frac{E}{R}
\]

\[
i(t) = \frac{E}{R} \left( 1 - e^{-\frac{E}{R} t} \right)
\]

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2. **Analytical Solution of a Difference Equation**

Using trapezoidal on the L

\[
\frac{\Delta t}{2 + \Delta t/T} \left( \frac{B}{A} \right) i(t) = \frac{B}{2 + \Delta t/T} \frac{\Delta t/L}{2 + \Delta t/T} U(t) + \frac{\Delta t/L}{2 + \Delta t/T} U(t - \Delta t)
\]

\[T = \frac{L}{R} = \text{time constant}\]

Rewriting as a difference equation on \(i(t)\), for \(U(t)\) as input,

\[i(t) - A i(t - \Delta t) = B U(t) + B U(t - \Delta t)\]

Following the well-known solution procedure of the solution of a differential equation:

Total Solution = Transient + Steady State
Solution of a Difference Equation (cont.)

a) Transient

From equation with no forcing function (homogeneous equation)

\[ l(t) - A i(t - \Delta t) = 0 \]

Let \( K = \) solution step number, e.i.,

\[ t = K \Delta t \]

\( \{ t = 0, \Delta t, 2\Delta t, \ldots \} \)

In terms of the solution step,

\[ l(k) - A i(k-1) = 0 \]

homogeneous equation

Assume solution has the form \( l(k) = p^k \)

\[ p^k - Ap^{k-1} = 0 \]

Factoring out \( p^k \),

\[ 1 - Ap^{-1} = 0 \]

\( P = A = \frac{2 - \Delta t / T}{2 + \Delta t / T} \)

\[ \boxed{l_h(k) = C P^k} \]

Transient Solution

constant (from initial conditions)
Solution of a Difference Equation (Cont. 2)

b) Steady State

Directly from the circuit,
\[ \text{iss}(k) = \frac{E}{R} \]

Steady-State Solution

c) Complete Solution

\[ i(k) = i_h(k) + \text{iss}(k) = C P^k + \frac{E}{R} \]

System Stable
\[ \Rightarrow P < 1 \]

Initial Conditions:
\[ i(0) = 0 \] for \( k = 0 \) \[ \Rightarrow i(0) = C + \frac{E}{R} \], \( C = -\frac{E}{R} \)

\[ i(k) = \frac{E}{R} (1 - P^k) \]

\[ P = \frac{2 - \Delta t / T}{2 + \Delta t / T} \]

\[ K = \frac{t}{\Delta t} \]

Discrete-Time Solution for Trapezoidal

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3. STABILITY OF THE TRANSIENT SOLUTION

Continuous-Time System:

Transient Solution:
\[ i(t) = \sum_{n=1}^{N} C_n e^{-p_n t} \]

STABLE \( \Rightarrow \) \( p_1, p_2, \ldots, p_n > 0 \) or POLES \( \leq 0 \).

Transfer Function:
\[ H(s) = \frac{(s+g_1)(s+g_2)\cdots(s+g_n)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \]

poles give the exponential constants

Discrete-Time System:

Transient Solution:
\[ i(k) = \sum_{n=1}^{N} C_n p_n^k \]

STABLE \( \Rightarrow \) \( |p_1|, |p_2|, \ldots, |p_n| < 1 \)

Transfer Function:
\[ H(z) = \frac{(z+g_1)(z+g_2)\cdots(z+g_n)}{(z+p_1)(z+p_2)\cdots(z+p_n)} \]

\( z \)-domain transfer function

\( z \)-poles give the power terms

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4. Z-TRANSFORM

Provides compact notation for analysis of discrete-time systems.
Analogous to Fourier or bilateral Laplace in continuous-time systems.

Continuous Time:
\[ Y(t) \xrightarrow{\mathcal{F}_s} \mathcal{L}^{-s} Y(s) \quad s = j\omega \text{ for Fourier} \]

Discrete Time:
\[ y(t-n\Delta t) \xrightarrow{\mathcal{Z}} z^{-n} Y(z) \quad z = \text{complex variable} \]

Frequency Response from Z-Transform:
Let \( n=1 \) and \( s = j\omega \),
\[ y(t-n\Delta t) \xrightarrow{\mathcal{F}} e^{-jn\omega t} Y(\omega) \]
\[ y(t-n\Delta t) \xrightarrow{\mathcal{Z}} Z^{-1} Y(z) \]

If we make \( z = e^{j\omega t} \), we get the frequency response of the discrete-time system.
5. TRANSFER FUNCTION IN Z-DOMAIN

Inductance discretized with trapezoidal:

\[ i(t) - i(t-\Delta t) = \frac{\Delta t}{2L} u(t) + \frac{\Delta t}{2L} u(t-\Delta t) \]

Applying Z-Transform:

\[ I(z) - z^{-1} I(z) = \frac{\Delta t}{2L} V(z) + \frac{\Delta t}{2L} z^{-1} V(z) \]

\[ \frac{I(z)}{V(z)} = Y_e(z) = \frac{\frac{\Delta t}{2L}}{z - 1} \quad \text{Transfer function in Z-Domain} \]

By making \( z = e^{j\omega t} \) we get the frequency response of the discretized:

\[ Y_e(\omega) = \frac{\frac{\Delta t}{2L}}{e^{j\omega t} - 1} \quad \text{Result obtained earlier for frequency response} \]
6. **Z-Domain Transfer Function of Integration Rules**

\[
\frac{L(t)}{V(t)} \quad Y_e(s) = \frac{I(s)}{V(s)} = \text{integrator}
\]

**Rule**

**Trapezoidal:** \( Y_e(s) = \left( \frac{\Delta t}{L} \right) \frac{1}{2} \frac{3+1}{s-1} \)

**Backward Euler:** \( Y_e(s) = \left( \frac{\Delta t}{L} \right) \frac{3}{s-1} \)

**Simpson:** \( Y_e(s) = \left( \frac{\Delta t}{L} \right) \frac{3^2 + 4\cdot3 + 1}{3^2 - 1} \)

**Gear 2nd Order:** \( Y_e(s) = \left( \frac{\Delta t}{L} \right) \frac{2s^2}{3s^2 - 4s + 1} \)

**Poles & zeroes**

- \( p_1 = 1 \) Stable Integrator
- \( z_1 = -1 \) Stable Differentiator *
- \( p_1 = 1 \) Stable Integrator
- \( z_1 = 0 \) Stable Differentiator **
- \( p_1 = 1 \) Stable Integrator *
- \( p_2 = -1 \)
- \( z_1 = -0.268 \) Stable Differentiator
- \( z_2 = -3.732 \) Unstable Differentiator
- \( p_1 = 1 \) Stable Integrator
- \( p_2 = 0.93 \)
- \( z_1 = 0 \) Stable Differentiator **
- \( z_2 = 0 \)

* Bounded oscillations at discontinuities

** Critical damping

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A step current into an inductance is a discontinuity because physically current cannot change instantly in an L (the same applies to a step voltage into a capacitance).

During simulation, this is a common situation during operation of ideal switches or ideal power electronics components.

**Continuous-Time Solution**

Even though not physically possible, the problem is defined analytically:

\[ V = L \frac{di}{dt}, \quad i(t) = U(t) \]

\[ \Rightarrow V(t) = L \delta(t) \] impulse function

\[ U(t) \] discrete

\[ \delta(t) \] discrete
Discontinuities (Cont.)

**DISCRETE-TIME SOLUTIONS**

**Trapezoidal** \( u(t) = -u(t-\Delta t) + \frac{2L}{\Delta t} i(t) - \frac{2L}{\Delta t} i(t-\Delta t) \)

**Backward Euler** \( u(t) = \frac{L}{\Delta t} i(t) - \frac{L}{\Delta t} i(t-\Delta t) \)

---

### Trapezoidal

<table>
<thead>
<tr>
<th>( t )</th>
<th>( i(t) )</th>
<th>( u(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>1</td>
<td>2L/\Delta t</td>
</tr>
<tr>
<td>2( \Delta t )</td>
<td>1</td>
<td>-2L/\Delta t</td>
</tr>
<tr>
<td>3( \Delta t )</td>
<td>1</td>
<td>2L/\Delta t</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

### Backward Euler

<table>
<thead>
<tr>
<th>( t )</th>
<th>( i(t) )</th>
<th>( u(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>1</td>
<td>L/\Delta t</td>
</tr>
<tr>
<td>2( \Delta t )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3( \Delta t )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

---

Sustained oscillations, Ave. ok

---

No oscillations. Correct area for the impulse.

---

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8. CDA PROCEDURE

**Trapezoidal with CDA**

<table>
<thead>
<tr>
<th>t</th>
<th>i(t)</th>
<th>U(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BE</td>
<td>Δt/2</td>
<td>1</td>
</tr>
<tr>
<td>BE</td>
<td>Δt</td>
<td>1</td>
</tr>
<tr>
<td>Trap</td>
<td>2Δt</td>
<td>1</td>
</tr>
<tr>
<td>Trap</td>
<td>3Δt</td>
<td>1</td>
</tr>
</tbody>
</table>

**Transient Simulation with CDA**

1. Use trapezoidal normally.
2. If a discontinuity occurs, change to backward Euler and perform two Δt/2 solution steps.
3. Go back to trapezoidal until next discontinuity.

**CDA**

\[ U(t) = \frac{v}{Δt} \]

\[ \text{area} = L \]

**When to use CDA?**

1. Ideal-switch operation.
2. Ideal diodes, thyristors, etc.
3. When changing regions in piecewise linear elements.
4. In synchronous machine model.

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CDA Procedure (Cont.)

\[ i(t) \quad L \quad U(t) \quad \Rightarrow \quad i(t) \quad R \quad C \quad \frac{C}{\Delta t} \quad \frac{C}{\Delta t} \quad \frac{C}{\Delta t} \quad V(t) \]

Trapezoidal with \( \Delta t \):

\[ R = \frac{2L}{\Delta t} \quad C \frac{C}{\Delta t} = -V(t - \Delta t) - \frac{C}{\Delta t} i(t - \Delta t) \]

Backward Euler with \( \Delta t \):

\[ R = \frac{L}{\Delta t} \quad C \frac{C}{\Delta t} = -\frac{L}{\Delta t} i(t - \Delta t) \]

Backward Euler with \( \Delta t/2 \) (for CDA):

\[ R = \frac{L}{\Delta t/2} = \frac{2L}{\Delta t} \quad C \frac{C}{\Delta t} = -\frac{2L}{\Delta t} i(t - \frac{\Delta t}{2}) \]

Same as for trapezoidal!

\[ \Rightarrow \text{Network } [C] \text{ matrix does not change. Only history formula changes.} \]
Results with and without CDAs are shown on next two pages.

CDA

- No damping resistances needed across inductances with

symmetric during non-conduction.

\[ R = 10 \, \Omega \]

\( R = 10 \, \Omega \) in parallel with diodes, to keep voltages

- Without snubber circuits, connect large resistances (e.g.:

- With CDAs, snubber circuits are not needed, if represented,

\[ R = 10 \, \Omega \]

1-phase diode bridge rectifier from N. Mohan, „Computer

Exercises for Power Electronics Studies“, 1990.
1-PHASE DIODE BRIDGE RECTIFIER
Voltages. Scale: $10^{\times}(2)$

Time scale: $10^{\times}(-1)$ s.
Solution method remains linear, when flux becomes larger than knee, solution switches from unsaturated linear slope to saturated slope. 

- Nonlinear inductance in DC/EPRI EMT and ATP.
- Generalization to more than two slopes is the "type 98" transformers.
- Approximation for nonlinear magnetizing inductances of such a two-slope inductance is often a reasonable approximation for a piecewise linear inductance by:

\[ L_1 \begin{cases} \frac{\partial}{\partial \phi} = p_1 \frac{\partial}{\partial \phi} = p_2 \end{cases} \]

The nonlinear element is approximated by a piecewise linear elements method.

- Solution of nonlinear elements with the composition of nonlinear representations.

The EMT uses two methods for nonlinear elements:
Effect exaggerated in both figures.

\[ L_2 \text{-slope after temporary overshoot:} \]

MicroTran Version 2.06 forces solution back onto specified.

\[ \gamma \]

Specified curve

Overshoot

Point on: discovered to be above \( \gamma \) knee. It follows \( L_2 \)-slope from that.

Most EMTP versions have an "overshoot" when \( \gamma \) is