WAVE PROPAGATION IN IDEAL LINES

THE CP-LINE MODEL
1. WAVE PROPAGATION IN IDEAL LINES

\[ R = \sigma / \mu, \quad L = H / \mu, \quad G = S / \mu, \quad C = F / \mu \]

\[ V_2 - V_1 = - (R \Delta x) i \quad - (L \Delta x) \frac{\partial i}{\partial t}; \quad i = i_1 \]

\[ i_2 - i_1 = - (G \Delta x) V \quad - (C \Delta x) \frac{\partial V}{\partial t}; \quad V = V_2 \]

\[ \Delta x \rightarrow dx \]

\[ - \frac{\partial V}{\partial x} = Ri + L \frac{\partial i}{\partial t} \]

\[ - \frac{\partial i}{\partial x} = GV + C \frac{\partial V}{\partial t} \]

\[ \text{Line equations for} \]

\[ \text{constant parameters} \quad R, L, G, C \]

\[ \text{Some values for power lines:} \]

\[ L = 1 \text{mH/km} \]

\[ C = 10 \text{mF/km} \]

\[ R = 0.01 \text{S/km (60 Hz)} \]

\[ R^0 = 0.3 \text{S/km (60 Hz)} \]

\[ (\text{In reality } R = R(\omega), \quad L = L(\omega) \]

\[ G = \text{small, } C = \text{constant} \]

\[ \text{for overhead transmission lines} \]

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2. IDEAL LINE WITH NO LOSSES

Assume $R=0$, $G=0$, $L=\text{constant}$, $C=\text{constant}$

$$-\frac{\partial u}{\partial x} = L \frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial x} = C \frac{\partial u}{\partial t}$$

Since there is no damping, try "steady-wave" solution:

$$V = Z_c i$$

$$-Z_c \frac{\partial i}{\partial x} = L \frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial x} = Z_c C \frac{\partial i}{\partial t}$$

Dividing:

$$Z_c = \pm \sqrt{\frac{L}{C}}$$

Voltage and current waves travel down the line without changing shape.

Shape of voltage wave identical to shape of current wave.
Velocity of Propagation

From the voltage drop equation,
\[ V_2 - V_1 = -(L \Delta x) \frac{\partial i_1}{\partial t} ; \quad V_2 = Z_c I_2, \quad V_1 = Z_c I_1 \]

Dividing over \( Z_c \)

\[ \frac{I_2}{I_1} = -\left(\frac{L}{Z_c} \right) \frac{\partial i_1}{\partial t} \]

Making \( \partial i_1/\partial t \) finite:

\[ I_2(t) - I_1(t) = -\frac{1}{Z_c} \Delta x \frac{i_1(t) - i_1(t-\Delta t)}{\Delta t} \]

If the wave propagated intact,

\[ i_2(t) = i_1(t-\Delta t), \quad \text{and} \]

\[ a = \frac{\Delta x}{\Delta t} = \frac{Z_c}{L}, \quad \text{or} \]

\[ a = \sqrt{\frac{1}{LC}} \quad \text{Velocity of Propagation} \]

- For a single conductor above ideal ground \( a \approx 300 \text{ km/s} \) (speed of light) and \( Z_c \approx 500 \Omega \).
- For oil-impregnated cable \( a \approx 150 \text{ km/s} \) and \( Z_c = 50 \Omega \).

- In non-dispersive medium (no losses)

\[ a = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{\mu e}} \quad \mu = \text{permeability} \quad \mu_0 = 4\pi \times 10^{-7} \]

\[ e_0 = 8.854 \times 10^{-12} \]

\[ a_0 = \sqrt{\frac{1}{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}} = 2.998 \times 10^8 \text{ m/s} \approx 300 \text{ km/s} \]
Direction of Current Flow

\[ V_{b} \approx V_{f} \quad \Rightarrow \quad V_{x} = V_{f} - V_{b} \]

\[ V_{b} \approx V_{f} \quad \Rightarrow \quad V_{x} = V_{f} + V_{b} \]

\[ i_{b} = \frac{V_{f}}{Z_{c}} \quad i_{b} = -\frac{V_{b}}{Z_{c}} \]

If \( V_{b} \) is positive \( \Rightarrow i_{b} \) is negative \( \Rightarrow \) flows to the left

If \( V_{b} \) is negative \( \Rightarrow i_{b} \) is positive \( \Rightarrow \) flows to the right (even though it travels to the left!)

\[ l_{b} < 0 \quad \Rightarrow \text{flows to the left} \]

\[ l_{b} > 0 \quad \Rightarrow \text{flows to the right} \]

Note that even though \( i_{b} \) travels to the left, \( l_{b} \) flows to the right!
ELEMENTARY WAVES (Summary)

\[ V_f = Z_c i_f \]
\[ V_b = -Z_c i_b \]

\[ i = 0 \] (\(+\) goes back, \(-\) bounces back)

\[ U = 0 \] (\(+\) going down, \(-\) going up)
Details:

**Line Terminated in** $R = Z_c$

Voltage wave "sees" $Z_c$ as it travels down the line. When it reaches $m$, it still sees $Z_c$. It continues through until it's gone!

Step voltage will "fill" the line at the step value.

Where did the energy go? (it dissipates in $R = Z_c$)
3. FORWARD AND BACKWARD WAVES

Two solutions satisfy the line equations

\[ \begin{align*}
V_f &= Z_c i_f \text{ wave } + x \\
V_b &= -Z_c i_b \text{ wave } - x
\end{align*} \]

General solution is then

\[ \begin{align*}
V &= V_f + V_b \\
i &= i_f + i_b
\end{align*} \] General Solution to Line Equations

Since at open end \( m \), \( i_m = 0 \)

\[ i_m = i_f + i_b = 0 \quad \text{or} \quad i_b = -i_f \]

From which \( V_b = -Z_c i_b = -Z_c (-i_f) = V_f \) is generated.

\[ \begin{array}{c}
V_m = V_f + V_b \\
V_m = 2 V_f \\
i_m = 0
\end{array} \]
Shorted Line

At node m,

\[ V_m = V_f + V_b = 0 \Rightarrow V_b = -V_f \]

\[ i_b = -\frac{V_b}{Z_c} = -\frac{-V_f}{Z_c} = i_f \]

\[ I_m = i_f m + i_b m = 2 i_f = 2 \frac{E}{Z_c} \]
Line Terminated in $R \neq Z_c$

\[ \frac{V_f}{i_f} = Z_c \quad \text{but} \quad \frac{V_m}{i_m} = R \Rightarrow \text{new waves: } V_b \text{ and } i_b \text{ are generated} \]

\[ V_m = V_f + V_b \]

\[ i_m = i_f + i_b \]

\[ \text{with } i_f = \frac{V_f}{Z_c}, \quad i_b = -\frac{V_b}{Z_c} \quad ; \quad V_f + V_b = \frac{R}{Z_c} V_f - \frac{R}{Z_c} V_b \]

\[ V_b = \left(\frac{R-Z_c}{R+Z_c}\right) V_f = \rho V_f \]

\[ i_b = -\frac{V_b}{Z_c} = -\rho i_f \]

Reflection Coefficient for voltage wave

\[ R > Z_c \quad (\text{e.g., open circuit}) \Rightarrow \rho > 0 \]

\[ R < Z_c \quad (\text{e.g., short circuit}) \Rightarrow \rho < 0 \]

\[ V_m = V_f + V_b = (1 + \rho) V_f = 2 \frac{R}{R+Z_c} V_f \]
Line-Cable Energization

When $U_f$, reaches $m$, it sees an impedance equal to $Z_2$ (larger than $Z_1$)

$$U_{b1} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) U_f = \left(\frac{500 - 50}{550}\right) U_f = 0.82 E$$

The voltage applied to the line is

$$U_{f2} = (1 + 0.82) E = 1.82 E$$

If $m$ is open, after $U_{f2}$ arrives:

$$U_n = 2 \times 1.82 E = 3.64 E$$

(not just twice but almost four times!)

* We should always energize from the high $Z_C$ side so that we encounter a lower $Z_C$ at the junction *

* If energized from cable (low $Z_C$)
When $U_f_2$ reaches $m$, it sees an impedance equal to $Z_1$

$$U_b_2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right) U_f_2 = \left(\frac{50 - 500}{50 + 500}\right) U_f_2 = -0.82 E$$

The voltage applied to the cable is

$$U_f_1 = (1 - 0.82) E = 0.18 E$$

If $K$ is open, after $U_f$ arrives:

$$U_k = 2 \times 0.18 E = \boxed{0.36 E}$$

Even less than rated!
Lightning Discharges

E.g., $i_f = 25 \text{ kA}$ (90% probability)

\[ i_f = \frac{I_0}{2}, \quad I_0 = \text{Tables} \]

\[ i_f = 25 \text{ kA} \Rightarrow U_f = 25 \times 10^3 \times 2000 = 50 \text{ MV} \]

\[ P = \frac{500/2 - 2000}{500/2 + 2000} = -0.778 \quad U_f = U_f = (1 - 0.778)50 \text{ MV} = 11.1 \text{ MV} \]

Lightning Channel:

\[ Z_c \approx 1000 \Omega \text{ to } 3000 \Omega \]
Thevenin Wave

\[ A \rightarrow U_{f1} \]

\[ 1 \quad Z_{c1} \]

\[ 2 \quad Z_c \]

\[ 3 \]

\[ Z_c \]

\[ Z_{TH} \]

\[ \pm E_{TH} \]

\[ \text{reference} \]

\[ \pm 2U_{f1} \]

\[ V \text{ open-circuit at } K = \text{ voltage at } K \text{ with lines } 2 \text{ and } 3 \text{ not connected} \]

\[ = 2U_{f1} \]

\[ Z_{TH} = \text{Impedance "looking" into } K \text{ with lines } 2 \text{ and } 3 \text{ not connected} \]

\[ = Z_{c1} \]

Connecting 2 and 3 to the Thevenin:

\[ U_K = 2U_{f1} \times \frac{Z_{c1}/2}{Z_{c1} + Z_{c1}/2} = 2 \times 50 \text{ HV} \times \frac{250}{2000 + 250} = 11.1 \text{ HV} \]

as before!
Norton Form:

\[ Z_{th} = a \text{ as before} = Z_{c1} \]
\[ i_N = I \text{ short-circuit at } K = \text{ current at } K \text{ with } K \text{ shorted and lines } 2 \text{ and } 3 \text{ not connected.} \]

In the Example:

\[ 2I_{f1} = I_0 \text{ in Tables} \]
\[ \text{Tables give directly current source to use in EMTP.} \]

Since \( Z_{c1} = 2000 \Omega \text{ or so}, \) it is normally neglected compared to \( Z_c \) of the lines and circuit is usually reduced to:

\[ V_K = I_0 \times \frac{Z_c}{2} \]
\[ = 50 \times 250 \Omega = 12.5 \text{ MV} \]

(\text{It was 11.1 MV taking into account } Z_{c1} = 2000 \Omega \text{ of the Lightning channel.})
1. CONSTANT-PARAMETERS LINE MODEL

Based on ideal line:

\[ R = 0, G = 0; \quad L, C \text{ constant} \]

\[ \frac{d}{dx} \begin{cases} \dot{V}(x,t) \\ \dot{I}(x,t) \end{cases} = \frac{1}{LC} \begin{cases} V(x,t) \\ I(x,t) \end{cases} \]

**General Solution**

\[ V = V_f + V_b \quad \text{or} \quad V(x,t) = V_f(x,t) + V_b(x,t) \]

\[ I = i_f + i_b \quad \text{or} \quad I(x,t) = i_f(x,t) + i_b(x,t) \]

\[ V_f = Z_c i_f \quad \text{or} \quad V(x,t) = V_f(x,t) \]

\[ V_b = -Z_c i_b \quad \text{or} \quad V(x,t) = V_b(x,t) \]

**Characteristic Impedance**

\[ Z_c = \frac{1}{\sqrt{LC}} \]

**Velocity of Propagation**

\[ \alpha = \sqrt{\frac{1}{LC}} \]

**Forward Wave** \((x - at)\) = function shifting on \(x\)-axis by amount "at" to the right

**Backward Wave** \((x + at)\) = function shifting on \(x\)-axis by amount "at" to the left

\[ V(x,t) = V_f(x-at) + V_b(x+at) \]

\[ I(x,t) = \frac{1}{Z_c} V_f(x-at) - \frac{1}{Z_c} V_b(x+at) \]

_D'Alembert 1747_

\[ f(x) \quad \text{or} \quad f(x-\alpha) \]
2. EQUIVALENT CIRCUIT FOR EMTP

Simple trick! Combine $U + Z_c i$

$U(x,t) + Z_c i(x,t) = 2 U_f (x-\alpha t)$

forward "PERTURBATION WAVE". It suffers no reflection regardless of termination.

Suppose it starts at $k (x=0)$ at time $t_1$:

$U_k(t_1) + Z_c i_k(t_1) = 2 U_f (\alpha t_1)$

It arrives at $m (x=l)$ at time $t_2 = t_1 + \varepsilon = t_1 + l/\alpha$

$U_m(t_1+\varepsilon) + Z_c i_m(t_1+\varepsilon) = 2 U_f (l - \alpha (t_1+\varepsilon))$

$= 2 U_f (-\alpha t_1)$

That is, with $t_1 + \varepsilon = t$,

$U_m(t) + Z_c i_m(t) = \frac{U_k(t-\varepsilon) + Z_c i_k(t-\varepsilon)}{\text{present values}}$

$\underbrace{\text{history}}$
Equivalent Circuit at Line Ends

Dommel’s Equivalent Circuit

\[ i_k(t) \rightarrow Z_c \]
\[ V_k(t) \]
\[ E_{kh}(t) \]
\[ E_{mh}(t) \]
\[ V_m(t) \]
\[ Z_c \]
\[ l_m(t) \]

\[ E_{kh}(t) = V_m(t-\tau) + Z_c l_m(t-\tau) \]
\[ E_{mh}(t) = V_k(t-\tau) + Z_c i_k(t-\tau) \]

- Circuits have same form as for L’s & C’s
- Nodes k and m are decoupled
- No integration rule was used
- Exact except for truncation errors

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Interpolation Errors

\[ \Delta t = 0.02 \text{ms} \]

At \( t = 0.16 \text{ms} \)

\[ E_{\text{wh}}(t) = V_k(t-\varepsilon) + Z_c i_k(t-\varepsilon) \]

\[ E_{\text{wh}}(0.16) = V_k(0.16-0.10) + Z_c i_k(0.16-0.10) \]

\[ = V_k(0.06) + Z_c i_k(0.06) \]

Exact Solution!

\[ \Delta t = \frac{\varepsilon}{n} \quad \text{n integer} \Rightarrow \text{Exact Solution} \]

Most cases not possible to make \( \Delta t \) submultiple of all \( \varepsilon \)'s.
Interpolation Errors (cont.)

\[ E_{mk}(t) = V_k(t - \tau) + Z_c l_k(t - \tau) \]

\[ E_{mk}(0.18) = V_k(0.18 - 0.10) + Z_c l_k(0.18 - 0.10) \]

\[ = V_k(0.08) + Z_c l_k(0.08) \]

\[ \text{Not in Table} \]

Error Criterion:

\[ \Delta t < \frac{1}{5} t_a \text{ to } \frac{1}{10} t_a \text{ of } \tau \]

to minimize interpolation errors
3. **CP-LINE MODEL**

\[
\begin{align*}
Rt/4 & \quad \text{ideal} & \quad Rt/2 & \quad \text{ideal} & \quad Rt/4 & \quad \text{m} \\
\text{k}\Omega & \quad \text{at} & \quad l/2 & \quad \text{k}\Omega & \quad l/2 & \quad \text{k}\Omega \\
\end{align*}
\]

\[Rt = R \times l \quad \text{(km)}\]

- Losses are lumped and split into three parts.
- Reasonable approximation if \(Rt < \frac{\lambda}{4}\) \(R_{\text{total}} < \frac{\lambda}{10}\) \(\text{(More recent criteria by Dr. Durrant)}\)
- For lightning studies, e.g., \(f=500\ \text{kHz}\), condition \(Rt < \frac{\lambda}{4}\) may not be true. (Suggestion: split line into two).
- In actual line, parameters are frequency dependent and \(R\) is distributed.