BACKGROUND
ON ELECTRIC TRANSIENT PROGRAMS
Pioneering Work

- L.W. Nagel, R. Rohrer, Pederson (Berkeley)  
  CANCER ("Computer Analysis of Nonlinear Circuits Excluding Radiation")  
  SPICE2  
  1975

- H.W. Domwel (B.P.A.)  
  EMT  
  1969
**ELECTRIC CIRCUITS SOLUTION**

**Phasor Analysis**

\[ E_s = E_s \angle \Theta_s \]
\[ E_s(t) = \sqrt{2} E_s \sin(\omega t + \Theta_s) \]
\[ I = \frac{E_s}{Z_t + Z_L} = I \angle \Theta_i \]
\[ i(t) = \sqrt{2} I \sin(\omega t + \Theta i) \]

**Transient Analysis**

\[ Ri + L \frac{di}{dt} = e_s(t) \]
\[ i(t) = \frac{E}{R} \left[ 1 - e^{-\frac{t}{T}} \right] \]
\[ T = \frac{L}{R} \]

**EMTP, SPICE ANALYSIS**

\[ i(t) = \frac{E_s(t) - E_L(t)}{R + 2L/\Delta t} \]
\[ E_L(t) = -V_L(t-\Delta t) - \frac{2L}{\Delta t} i(t-\Delta t) \]
\[ \Delta t = \text{time step size} \]
Analytical Solution of Differential Eqns.

- Very difficult for large networks
- Very difficult with nonlinearities
- Very difficult with diodes, thyristors, etc.
- Very difficult with frequency dependent components (skin effect)
# Step by Step Solution

\[ l_{1}(t) \rightarrow e(t) \rightarrow v_{2}(t) \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( e(t) )</th>
<th>( l_{1}(t) )</th>
<th>( v_{2}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50 ( \mu )s</td>
<td>2.5</td>
<td>0.9</td>
<td>1.2</td>
</tr>
<tr>
<td>100 ( \mu )s</td>
<td>3.2</td>
<td>1.2</td>
<td>1.5</td>
</tr>
<tr>
<td>150 ( \mu )s</td>
<td>3.7</td>
<td>1.3</td>
<td>1.4</td>
</tr>
<tr>
<td>200 ( \mu )s</td>
<td>3.9</td>
<td>1.3</td>
<td>1.6</td>
</tr>
<tr>
<td>250 ( \mu )s</td>
<td>4.2</td>
<td>1.2</td>
<td>1.6</td>
</tr>
<tr>
<td>300 ( \mu )s</td>
<td>4.1</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

\( \Delta t \)

-4-
MODELING OF
BASIC R, L, C
COMPONENTS
DISCRETIZATION OF R, L, C

**Resistance R**

\[ V(t) = RI(t) \]

E.g. \( R = 5 \Omega \)

\[ i(10 \text{ms}) = 30 \text{A} \Rightarrow V(10 \text{ms}) = 5 \times 30 = 150 \text{V} \]

**Inductance L**

\[ V(t) = L \frac{di(t)}{dt} \]

E.g. \( L = 2 \text{mH} \)

\[ i(10 \text{ms}) = 30 \text{A} \Rightarrow V(10 \text{ms}) = \text{??} \]

**Discretization**

\[ \int_{t}^{t} V(t) \, dt = L \int_{t}^{t} \frac{di(t)}{dt} \, dt = L \int_{t}^{t} di = L \left[ i(t) - i(t - \Delta t) \right] \]
<table>
<thead>
<tr>
<th>$t$</th>
<th>$i(t)$</th>
<th>$V(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>0.1</td>
<td>0.15</td>
</tr>
<tr>
<td>$2\Delta t$</td>
<td>0.2</td>
<td>0.35</td>
</tr>
<tr>
<td>$t-\Delta t$</td>
<td>0.8</td>
<td>0.45</td>
</tr>
<tr>
<td>$t$</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Want to solve at $t = t$

We know solution up to here

**Trapezoidal:**

$$\text{area} = \frac{V(t)+V(t-\Delta t)}{2} \cdot \Delta t$$

**Backward Euler:**

$$\text{area} = V(t) \cdot \Delta t$$

---

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3. TRAPEZOIDAL DISCRETIZATION

\[
\int_{t-\Delta t}^{t} V(t)dt = L \left[ i(t) - i(t-\Delta t) \right]
\]

\[
\frac{U(t) + U(t-\Delta t)}{2} \Delta t = L \frac{\dot{i}(t)}{\Delta t} - L \frac{\dot{i}(t-\Delta t)}{\Delta t}
\]

\[
U(t) = \left( \frac{2L}{\Delta t} \right) \frac{\dot{i}(t)}{\Delta t} + \left[ -U(t-\Delta t) - \frac{2L}{\Delta t} \frac{\dot{i}(t-\Delta t)}{\Delta t} \right]
\]

(equiv. resistance)

\[
U(t) = \frac{2L}{\Delta t} \frac{\dot{i}(t)}{\Delta t} + \mathcal{E}_h(t)
\]

\[
\Delta t = 2 \text{ ms}
\]

\[
i(10 \text{ ms}) = 30 \text{ A} \quad \Rightarrow \quad U(10 \text{ ms}) = \frac{2L}{\Delta t} \times 30 + \left[ -U(8 \text{ ms}) - \frac{2L}{\Delta t} \frac{\dot{i}(8 \text{ ms})}{\Delta t} \right]
\]

\[\text{known number} \quad 0.35\]
**CAPACITANCE C**

Continuous-time C

\[
\frac{di(t)}{dt} = C \frac{dU(t)}{dt}
\]

\[
i(t) dt = C \, dU(t)
\]

\[
\int_{t-\Delta t}^{t} i(t) dt = C \left[ U(t) - U(t-\Delta t) \right]
\]

\[
U(t) = \left( \frac{\Delta t}{2C} \right) i(t) + \left[ U(t-\Delta t) + \frac{\Delta t}{2C} i(t-\Delta t) \right]
\]

\[
\text{equiv. resistance } \quad \text{Eh (history)}
\]

\[
U(t) = \frac{\Delta t}{2C} i(t) + \text{Eh}(t)
\]

Discrete-time C

\[
i(t) R = \frac{\Delta t}{2C} Eh(t) m
\]

\[
\int_{t-\Delta t}^{t} i(t) dt \approx \frac{i(t) + i(t-\Delta t)}{2} \cdot \Delta t
\]
TRAPEZOIDAL RULE

Continuous Time

Discrete Time

\[ R = \frac{2L}{\Delta t} \quad E_k(t) \]

\[ E_k(t) = -U(t-\Delta t) - \frac{2L}{\Delta t} i(t-\Delta t) \]

\[ R = \frac{\Delta t}{2C} \quad E_k(t) \]

\[ E_k(t) = U(t-\Delta t) + \frac{\Delta t}{2C} i(t-\Delta t) \]
4. BACKWARD EULER RULE

\[ R \quad \text{2} \quad \rightarrow \quad R \]

\[ L \quad \text{2} \quad \rightarrow \quad R = \frac{L}{\Delta t} \quad E_k(t) \]

\[ C \quad \text{2} \quad \rightarrow \quad R = \frac{\Delta t}{C} \quad E_k(t) \]

\[ \quad \text{Continuous time} \quad \quad \text{Discrete time} \]

\[ E_k(t) = -\frac{L}{\Delta t} i(t-\Delta t) \]

\[ E_k(t) = V(t-\Delta t) \]

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5. **Other Rules**

a) **Forward Euler**
   1. Unstable

b) **Simpson**
   1. Unstable

c) **Gear Second Order**
   1. Pretty good

**How to Compare Rules?**

1. Accuracy
2. Stability
### Accuracy & Stability

<table>
<thead>
<tr>
<th>Method</th>
<th>Accuracy</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoidal</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>Backward Euler</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Forward Euler</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>Simpson</td>
<td>✓✓</td>
<td>✗</td>
</tr>
<tr>
<td>Gear 2nd Order</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

- **Trapezoidal**: numerical oscillations at discontinuities
- **Backward Euler**: strong phase error
- **Gear 2nd Order**: less oscillations, less phase error.

**CDA Technique**: Trap. + BE

Accuracy of Trapezoidal + Stability of Backward Euler
7. SIZE OF $\Delta t$

Smaller $\Delta t \Rightarrow$ more accuracy

For sampling at $\Delta t$,

$$f_{\text{Nyquist}} = \frac{1}{2\Delta t}$$

For accuracy of integration rule,

$$f_{\text{max}} = \frac{1}{5} f_{\text{Ny}} \text{ for } 3\% \text{ error with trapezoidal}$$

$$f_{\text{max}} = \frac{1}{10\Delta t} \Rightarrow \Delta t = \frac{1}{10 \times f_{\text{max}}} \text{ 3\% error trap.}$$

EXAMPLE:
Simulate a transient up to 5 kHz with 3\% trapezoidal error.

$$f_{\text{max}} = 5 \times 10^3$$

$$\Delta t = \frac{1}{10 \times 5 \times 10^3} = 20 \mu s$$
SOLUTION

ALGORITHM
1. NETWORK SOLUTION

Nodal Equation

\[
\begin{bmatrix}
G
\end{bmatrix}
\begin{bmatrix}
V
\end{bmatrix}
= \begin{bmatrix}
h
\end{bmatrix}
\]

\[
\text{Known current sources (external + history)}
\]

\[
\begin{bmatrix}
1 & \left(\frac{1}{2C} + \frac{1}{R} + \frac{\Delta t}{2L} + \frac{2C}{\Delta t}\right) & -\frac{\Delta t}{R} & -\frac{2C}{\Delta t} & 0 \\
2 & -\frac{1}{R} & \frac{1}{R} & 0 & 0 & 0 \\
3 & -\frac{\Delta t}{2L} & 0 & \frac{\Delta t}{2L} & 0 & 0 \\
4 & -\frac{2C}{\Delta t} & 0 & 0 & \frac{2C}{\Delta t} & 0 \\
5 & 0 & 0 & 0 & -\frac{1}{2C} & \frac{1}{Z_c}
\end{bmatrix}
\begin{bmatrix}
V_1(t) \\
V_2(t) \\
V_3(t) \\
V_4(t) \\
V_5(t)
\end{bmatrix}
= \begin{bmatrix}
L_{s1} + h_{10} + h_{13} + h_{14} \\
0 \\
-h_{13} \\
-h_{14} \\
h_{50}
\end{bmatrix}
\]
One further comment to Chapter 15.5: Never use Cramer's rule to solve linear equations (at least not for \( N > 2 \) or 3). If you do, you'd get the following relative solution times:

<table>
<thead>
<tr>
<th>Number of equations</th>
<th>Gauss elimination</th>
<th>Cramer's rule straightforward</th>
<th>Cramer's rule with Laplace expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.5 s</td>
<td>1 min</td>
<td>4 s</td>
</tr>
<tr>
<td>10</td>
<td>10 s</td>
<td>85 days</td>
<td>4 min</td>
</tr>
<tr>
<td>20</td>
<td>1 min</td>
<td>10'' years</td>
<td>4.6 days</td>
</tr>
</tbody>
</table>

It may be old hat for you that you shouldn't use Cramer's rule. But it wasn't always known just a few years ago. H. H. Skilling, in his book "Electrical Engineering Circuits" (which is, otherwise, an excellent book), John Wiley and Sons, 1965, says on p. 276: "... These are linear equations that can be solved by any convenient means. The method of determinants is so much easier and neater than any other that it is highly recommended." Or from another book by G. T. Funk, "Short-circuits in ac systems" (again an excellent book otherwise), Oldenbourg Munich 1962 (in German) on p. 72: "... There are a number of methods available for the solution of linear systems. Generally known are the methods of substitution, ... These methods are well suited for 2 or 3 unknowns, but are not well suited for a larger number of unknowns. In such cases it is best to use the method of determinants..."

*The names which he lists are all variations of the Gauss or Gauss-Jordan elimination.

Typical solution times for solving \([Y][V] = [I]\)

- No. of nodes: 267
- No. of branches: 423
- No. of non-zero elements above diagonal after triangularization: 1015

Solution times on IBM 7040: 16.5 s (complex elements) Repeat solution (complex) 1.6 s

(c) H. V. Dommel, 1975
Elimination process "for repeat solutions" uses information contained in upper (and in the case of unsymmetric matrices also lower) triangular matrix to extend elimination process to right-hand side \([I]\) (downward operations) and to find solution \([V]\) by backsubstitution. Operations \(\propto N^2\) (same as for multiplying inverse \([Y]^{-1}\) with \([I]\)). Is done whenever we want to solve a new case with a new vector \([I]\) but unchanged \([Y]\).

### Table. Number of operations for solving a system of linear equations.

1. Full matrices


<table>
<thead>
<tr>
<th>Method</th>
<th>Process</th>
<th>Number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss elimination</td>
<td>for matrix</td>
<td>(\frac{N^3}{2} - \frac{N^2}{3})</td>
</tr>
<tr>
<td>(triangularization)</td>
<td>1 repeat solution</td>
<td>(\frac{N^2}{3} - \frac{N^2}{2} + \frac{N}{6})</td>
</tr>
<tr>
<td>Gauss elimination</td>
<td>for matrix</td>
<td>(\frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6})</td>
</tr>
<tr>
<td>(using scalar products)</td>
<td>1 repeat solution</td>
<td>(\frac{N^2}{3} - \frac{N^2}{2} + \frac{2N}{3})</td>
</tr>
<tr>
<td>Gauss-Jordan (diagonalization)</td>
<td>for matrix</td>
<td>(\frac{N^3}{2} - \frac{N^2}{2})</td>
</tr>
<tr>
<td></td>
<td>1 repeat solution</td>
<td>(\frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6})</td>
</tr>
</tbody>
</table>

2. Sparse matrices for network solutions

Original matrix: 127 nodes (diag.)
153 branches (off-diag.)

After triangularization:
229 branches

X's are original elements
0's are elements introduced by triangularization

Example of near-optimal ordering of network admittance matrix for network with 127 nodes (source: Bonneville Power Administration).
2. EXAMPLE SYSTEM

\[ i_1 \]

\[ L_1 \]

\[ R \]

\[ Z_c, z \]

\[ 4 \]

\[ C \]

\[ i_2 \]

\[ 2L_1 \Delta t \]

\[ R \]

\[ 3 \]

\[ V \]

\[ E_s(t) \]

\[ hL_1 \]

\[ hL_2 \]

\[ hL_3 \]

\[ hL_4 \]

\[ \frac{V_1(t-\Delta t)}{2L_1} \]

\[ V_2(t-\Delta t) + i_{12}(t-\Delta t) \]

\[ G \]

\[ [U(t)] \]

\[ [h(t)] \]

\[ R \]

\[ V \]

\[ E_s(t) \]

\[ R \]

\[ V \]

\[ E_s(t) \]

\[ hL_3 \]

\[ hL_4 \]

\[ \frac{V_4(t-\Delta t)}{2L_2} \]

\[ U_3(t-\Delta t) + i_{34}(t-\Delta t) \]

\[ U_4(t-\Delta t) + i_{40}(t-\Delta t) \]

\[ hL_3 = \frac{1}{Z_c} U_4(t-\Delta t) + i_{43}(t-\Delta t) \]

\[ hL_4 = \frac{1}{Z_c} U_3(t-\Delta t) + i_{34}(t-\Delta t) \]

Node Equations

At \( t = 0, \Delta t, 2\Delta t, \ldots \), solve for

\[ G \]

\[ [U(t)] \]

\[ [h(t)] \]

Conductances

Node voltages

Current sources

(history + external)

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Example System (Cont.)

The voltages of the nodes with voltage sources are not unknown but known and the corresponding variables should be moved to the right hand side of the modal equation.

\[
\begin{bmatrix}
G_{AA} & G_{AB} \\
G_{BA} & G_{BB}
\end{bmatrix}
\begin{bmatrix}
U_A \\
U_B
\end{bmatrix}
=
\begin{bmatrix}
-h_A \\
-h_B
\end{bmatrix}
\]

Suppose nodes B have voltage sources:

\[
\begin{bmatrix}
G_{AA} \\
G_{BA}
\end{bmatrix}
\begin{bmatrix}
U_A \\
U_B
\end{bmatrix}
=
\begin{bmatrix}
h_A
\end{bmatrix}
\]

\[
\left[ G_{AB} \right]_{3\times1}
\begin{bmatrix}
U_B
\end{bmatrix}_{1\times1}
= \left[ E_s(t) \right]
\]

In the Example

\[
\begin{bmatrix}
\frac{\alpha t}{2L_1} + \frac{2c}{s^2} + \frac{1}{R} \\
-\frac{1}{R}
\end{bmatrix}
\begin{bmatrix}
U_2(t) \\
U_3(t)
\end{bmatrix}
= \begin{bmatrix}
-h_{L1} + h_c \\
-h_{L3}
\end{bmatrix}
- \begin{bmatrix}
\frac{\alpha t}{2L_1} \\
0
\end{bmatrix}
E_s(t)
\]

\[
\begin{bmatrix}
0 \\
\frac{\alpha t}{2L_2} + \frac{1}{Z_c}
\end{bmatrix}
\begin{bmatrix}
U_4(t)
\end{bmatrix}
= \begin{bmatrix}
h_{L4} + h_{L2}
\end{bmatrix}
- \begin{bmatrix}
0
\end{bmatrix}
\]

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3. SOLUTION ALGORITHM

\[
\begin{bmatrix} G \end{bmatrix} \begin{bmatrix} V(t) \end{bmatrix} = \begin{bmatrix} I(t) \end{bmatrix}
\]

\[
\begin{bmatrix} G_{AA} \end{bmatrix} \begin{bmatrix} V_A(t) \end{bmatrix} = \begin{bmatrix} I_A(t) \end{bmatrix} - \begin{bmatrix} G_{AB} \end{bmatrix} \begin{bmatrix} V_B(t) \end{bmatrix}
\]

\text{Solution:}

\[
\begin{bmatrix} V_A(t) \end{bmatrix} = \begin{bmatrix} G_{AA} \end{bmatrix}^{-1} \begin{bmatrix} I_A(t) \end{bmatrix} - \begin{bmatrix} G_{AA} \end{bmatrix}^{-1} \begin{bmatrix} G_{AB} \end{bmatrix} \begin{bmatrix} V_B(t) \end{bmatrix}
\]

In EMTP \([G_{AA}]^{-1}\) is not explicitly evaluated. Gauss reduction and back-substitution are used instead.

Algorithm

\text{at } t = t

a) Evaluate current sources \([I_A(t)]\) from previous solution and external current sources.

b) Solve for node voltages \([V_A(t)]\)

c) Update history functions with current results.

d) Go back to a) for \(t + dt\).
Solution Algorithm (cont.)

Read-in Data

Build \([Y]\) matrix for steady-state solution
Build \([G]\) matrix for transient solution

Find steady-state solution to initialize histories for values at \(t \leq 0\)

\[ t = \Delta t \]

Evaluate Current & Voltage Sources

Solve for Node Voltages

Update History Functions

\[ t = t + \Delta t \]
4. SWITCHES

\[ i_{kw} \xrightarrow{T_{close}} \text{Ideal time switch} \]

\[ T_{close} \]

\[ 0 \at 2\Delta t \at 3\Delta t \at 4\Delta t \]

- MicroTran: Closes at $2\Delta t$ (nearest time step)
- ATP, DCG/EPRI: Closes at $3\Delta t$ (next time step)

\[ T_{open} \]

\[ i_{kw} \xrightarrow{T_{open \ circuit \ open}} \text{Program open at } 6\Delta t \]

- Change of direction detected at $5\Delta t$

a) After $T_{open}$, $i_{kw}$ is checked for change of direction.
b) Change of direction is detected after solution at $5\Delta t$.
c) Solution for $5\Delta t$ is performed with switch closed.
d) Solution for $6\Delta t$ is performed with switch open.
Switches (Cont.)

Close/Open Simulation

SPICE and Others:

\[ k \frac{10^{-5}}{m} \quad k \frac{10^{-4}}{m} \]

\[ k \]

EMTP

1 \( \begin{bmatrix} (g_{10} + g_{12} + g_{13}) & -g_{12} & -g_{13} \\ -g_{12} & (g_{20} + g_{12} + g_{23}) & -g_{23} \\ -g_{13} & -g_{23} & (g_{30} + g_{13} + g_{23}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \]

Open

\[ \begin{bmatrix} (g_{10} + g_{20} + g_{13} + g_{23}) & -(g_{13} + g_{23}) \\ -(g_{13} + g_{23}) & (g_{30} + g_{13} + g_{23}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = \begin{bmatrix} h_1 + h_2 \\ h_3 \end{bmatrix} \]

Closed

To close switch by collapsing nodes 1 and 2:

a) Retain node 1;
b) Add rows 1 & 2 in \([G]\) and in current source;
c) Add columns 1 & 2 in \([G]\).
**Sources**

**Type 14**

\[ V(t) = A \cos(\omega t + \phi) \]

- peak value
- degrees

**Type 11**

Step Function