Multiple–Differential Encoding for Multi–Hop Amplify–and–Forward IR–UWB Systems

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Abstract—In this paper, we propose a novel multi-hop relaying scheme to improve the performance and coverage of impulse-radio-based ultra-wideband (IR-UWB) systems. With regard to a simple practical realization, we focus on a non-coherent system setup in conjunction with amplify-and-forward (A&F) relaying. In particular, we propose to employ a multiple-differential encoding scheme at the source node and single differential decoding at each relay and at the destination node, respectively, so as to efficiently limit intersymbol-interference effects at the destination node. For a dual-hop system we derive a closed–form expression for the signal-to-noise ratio (SNR) at the destination node, and for the general multi-hop case we provide a simple recursive formula for SNR calculation. Based on these SNR results, we obtain a closed-form expression for the optimal transmit power allocation to the source node and the relay for a dual-hop system and a simple recursive suboptimal power allocation scheme for the multi-hop case, which permits a semi-distributed implementation with limited feedback between nodes. Simulation results illustrate the excellent performance of the proposed multiple-differential encoding scheme with A&F relaying for both uncoded and coded transmission compared to various alternative coherent and non-coherent schemes based on A&F relaying and decode-and-forward (D&F) relaying. Furthermore, our simulations confirm the (near-)optimal performance of the proposed power allocation solutions.

Index Terms—Ultra-wideband (UWB) communications, impulse radio (IR), amplify and forward (A&F) relaying, multiple hops, differential encoding, performance analysis, power allocation.

I. INTRODUCTION

ULTRA-WIDEBAND (UWB) radio is a wireless spectral underlay technology for transmitting signals with a bandwidth larger than 500 MHz or with a fractional bandwidth of more than 20% [1], [2], [3], [4]. Owing to their large bandwidth, UWB transmission techniques are, for example, envisioned for short-range high-speed indoor communications. Within the scope of this paper, focus is on impulse-radio (IR)-based UWB (IR-UWB) systems – such as the IEEE 802.15.4a standard [5] – where the transmitted signal consists of a train of pulses of very short duration [6]. Due to its simple practical realization and its robustness to multipath fading and intersymbol-interference (ISI) effects, IR-UWB has attracted considerable attention. Moreover, by employing pseudo-random time-hopping (TH) sequences, IR-UWB can also support multiuser communications with a minimum of multiple-access interference.

In order to limit interference to incumbent wireless services, the US Federal Communications Commission (FCC) has issued tight restrictions on the transmitted power spectral density (PSD) of UWB systems [7]. Because of these limitations, it is essential to capture at the receiver most of the signal energy provided by the large number of resolvable multipath components. A favorable property of IR-UWB systems is that they allow for an efficient energy combining at the receiver [8] – by means of either coherent Rake combining [9] or non-coherent energy detection schemes [10], [11]. Rake combining requires accurate channel estimation at the receiver and precise timing synchronization, which can be challenging in practice. In particular, a large number of “Rake fingers” is typically required in order to capture most of the signal energy [12]. As opposed to this, selective Rake (S-Rake) combiners collect only part of the signal energy, thus providing suboptimal performance at the benefit of a reduced receiver complexity. Still, accurate channel knowledge and precise timing synchronization are indispensable.

In contrast to this, non-coherent energy detection schemes relieve the receiver from any channel estimation task and are thus easier to realize in practice. They capture the energy of the multipath components by means of an autocorrelation of the received signal, followed by an integrate-and-dump (I&D) operation. Among these techniques, differential and transmitted-reference (TR) schemes are the most popular options [10]. In the TR scheme, pulses are transmitted in pairs, where the first pulse serves as reference pulse and the second pulse is modulated by the transmitted information bit sequence. The differential scheme, however, employs only single pulses, which are modulated by a differentially encoded information bit sequence. Therefore, a notable advantage of the differential scheme is that it is more energy-efficient and offers a higher data rate compared to the TR scheme [10], since the additional reference pulse is discarded. In the literature, several methods have been proposed to improve the performance of non-coherent IR-UWB schemes, such as reference filtering [13], weighted correlation [14], and multiple symbol detection [15].

In this paper, we investigate the use of relays as a means to overcome the limited coverage of IR-UWB systems [16], [17], [18], [19]. Instead of transmitting signals directly from a source node, $S$, to a destination node, $D$, the signal is
(in the simplest case) received by an intermediate relay, \(R_1\), which forwards the received signal from the source node to the destination node. In a more complicated multi-hop scenario consisting of \(m\) links, the source signal is forwarded via \((m-1)\) subsequent relays \(R_1, ..., R_{m-1}\). By this means, substantial path-loss gains can be achieved due to shorter link lengths. Two popular schemes for cooperative relaying are amplify-and-forward (A&F) and decode-and-forward (D&F) relaying, which were originally proposed for narrowband wireless channels [20]. Conventionally, A&F relays simply amplify and retransmit the received signal, whereas D&F relays first decode and then re-encode the received signal, before re-transmission is performed. Correspondingly, D&F relaying is usually more complex than A&F relaying, especially if a forward-error-correction (FEC) code is employed.

Previous work on relaying for UWB systems has focused on the ECMA-368 UWB standard [21], see [22], [23], space-time code design for coherent IR-UWB systems [17], coherent A&F and D&F relaying requiring a Rake combiner at each relay [18], [19], and dual-hop and multi-hop D&F relaying for IR-UWB with single-differential encoding at the source [16], [18]. Here, for the sake of a simple practical realization, we will focus on a combination of the differential IR-UWB scheme with A&F relaying. A straightforward combination of the two techniques with differential encoding at the source, simple A&F relaying at the intermediate relay(s), and differential decoding at the destination – which is well-known from narrowband systems [24], [25] – has the major drawback that the length of the effective overall channel impulse response (CIR) from source to destination increases with each hop. Compared to direct transmission (i.e., without any relaying), significantly larger guard intervals between the transmitted pulses are therefore required, in order to achieve a similar level of ISI. Otherwise, the increased amount of ISI will severely limit the overall performance and can even compromise the achieved path-loss gains. Note that an increased guard interval will significantly lower the effective data rate compared to direct transmission. As an alternative, we propose a multiple-differential encoding setup with \(m\)-fold differential encoding at the source in conjunction with single differential demodulation at each A&F relay and at the destination. By this means, the same level of ISI is achieved as in the case of direct transmission, without requiring an extended guard interval. To the best of our knowledge, such a use of multiple-differential encoding is novel and quite different from narrowband systems, where, e.g., double-differential encoding is employed to mitigate carrier frequency offsets [26]. Another significant advantage of our proposed A&F scheme with multiple-differential encoding is that we are able to devise analytical solutions for a (near) optimal power allocation between the source node and the relay(s). Similar results are not available for corresponding D&F-based schemes (cf. e.g. [16], [18]), especially not for the multi-hop case.

**Paper organization:** The remainder of the paper is organized as follows. In Section II, we describe the system setup under consideration, including the underlying channel model as well as the proposed structures of the source node, the relay node(s), and the destination node. In Section III, we first provide a thorough performance analysis of our proposed scheme in terms of the effective signal-to-noise ratio (SNR) at the destination node for the dual-hop case. Subsequently, we develop a recursive formula for calculating the effective SNR at the destination node for the multi-hop case. In Section IV, we derive a closed-form expression for the optimum transmit power allocation between source and relay node for dual-hop transmission, and present a near-optimal transmit power allocation strategy for the multi-hop case. Simulation results, which illustrate the excellent performance of our scheme, are presented in Section V. In particular, our system setup is compared with various alternative coherent and non-coherent schemes based on A&F and D&F relaying. Finally, Section VI concludes the paper.

**II. System Setup**

We consider a serial multi-hop differential IR-UWB system consisting of a source node \(S\), \((m-1)\) A&F relays \(R_1, ..., R_{m-1}\), and a destination node \(D\). Here, \(m > 1\) denotes the total number of hops. The link between source node \(S\) and relay \(R_1\) is in the sequel denoted as Link 1, the link between relay \(R_{i-1}\) and relay \(R_i\) is denoted as Link \(i (2 \leq i \leq m-1)\), and the link between the last relay \(R_{m-1}\) and the destination node \(D\) is denoted as Link \(m\). For the ease of exposition, we focus on the single-user case here. However, it is straightforward to extend our proposed multiple-differential A&F relaying scheme to the multi-user case, by incorporating corresponding TH sequences [5]. Before we provide a detailed description of the proposed multiple-differential A&F (MD-A&F) relaying scheme, we briefly recapitulate the IR-UWB transmission format and highlight the advantages of our scheme.

**A. IR-UWB Transmission Format**

In IR-UWB systems, the transmitted signal consists of a train of ultra-short pulses (on the order of nanoseconds), which are modulated by the transmitted information symbols [6]. Within the scope of this paper, focus will be on pulse-amplitude modulation (PAM). As illustrated in Fig. 1 (a), usually \(N_f > 1\) frames are used to convey a single information symbol (in the example, we have \(N_f = 2\)). By this means, the effective symbol energy at the receiver can be increased. However, this comes at the expense of a decreased throughput. After transmission, the IR-UWB signal is convolved with the UWB CIR, which can have a length on the order of hundreds of nanoseconds [4], depending on the radio environment under consideration. Correspondingly, the received pulses are considerably spread out in time, cf. Fig. 1 (b). In the differential setup, the energy of the underlying multipath signal components is collected at

\footnote{By ‘serial multi-hop’ we refer to a scenario, where the source node \(S\) and the A&F relays \(R_1, ..., R_{m-1}\) transmit subsequently, e.g., based on some global frame structure. Moreover, each relay \(R_i (i > 1)\) is assumed to process only signals received from relay \(R_{i-1}\), and the destination node \(D\) is assumed to process only signals received from the last relay \(R_{m-1}\). Correspondingly, performance advantages compared to direct transmission from the source node \(S\) to the destination node \(D\) will solely be based on path-loss gains. Diversity gains are not obtained by the considered relaying setup, since no diversity combining is performed at any node.}
the receiver by means of an I&D operation (for details see Sections II-D and II-E).

If a straightforward combination of the IR-UWB transmission format with A&F relaying is employed (e.g. similar to that used in narrowband systems [24], [25]), the effective CIR seen at the destination node results from a convolution of the UWB CIRs of all intermediate links. Correspondingly, the time spread of the received pulses will grow with each hop, leading to significant interpulse interference (IPI) and ISI, as the received pulses associated with one information symbol (i.e., with one set of $N_f$ frames) interfere with those associated with the subsequent symbol, see Fig. 1 (c). In order to keep the level of IPI and ISI comparable to the single-hop case, the only option is to increase the guard intervals between subsequent pulses and symbols, which will (further) decrease the effective throughput. In the following, we present the details of our proposed MD-A&F relaying scheme. A major advantage of our scheme is that the level of IPI and ISI is not increased compared to the single-hop case. In particular, since differential demodulation (in conjunction with an I&D operation) is performed at each relay, the time spread of the received pulses depends only on a single UWB CIR, so that the same guard interval size can be employed as in the case of a single hop.

B. Proposed Structure of Source Node

Fig. 2 shows the proposed structures of the source node $S$, the $i$th A&F relay $R_i$, and the destination node $D$. Throughout this paper, the employed relays $R_1, ..., R_{m-1}$ are assumed to have identical structures. We start with a description of the source node structure.

At the source node, the transmitted information bits, $b[n] \in \mathbb{Z}$, are encoded...
where \( q \) (the times differentially encoded according to \( w \) pulses) symbols \( q_{m+1} \) of \([5]\)). The coded bits will be on a convolutional encoding scheme (along the lines of \([5]\)). This paper we assume that relays are equipped with a single antenna only, used for both receiving and transmitting (in a half-duplex fashion). For the ease of exposition, however, the receiving branch and the transmitting branch of relay \( R_i \) are depicted separately here.

\[ g_r[k] = g_{r-1}[k]g_r[k-1] \quad 2 \leq r \leq m + 1, \quad (1) \]

where \( g_r[k] \in \{ \pm 1 \} \) denotes the \( k \)th intermediate symbol after the \((r-1)\)th differential encoder \((2 \leq r \leq m+1)\). The symbols \( q_{m+1} \) are then modulated onto a train of (real-valued) short pulses \( w_{tx}(t) \) with duration \( T_p \), according to \([3]\]

\[ s_1(t) = \sqrt{\alpha_1 E_g} \sum_{k=-\infty}^{\infty} \sum_{j=0}^{N_f-1} q_{m+1}[k] w_{tx}(t-jT_f-kT_s). \quad (2) \]

Here, \( s_1(t) \) denotes the signal transmitted by the source node, \( \alpha_1 \) the corresponding transmit power allocation factor, \( E_g \) the energy per pulse \((\int^{+\infty}_{-\infty} w_{tx}^2(t) \, dt = 1)\), \( N_f \) the number of frames used for conveying a single information symbol, \( T_f \) the frame duration, which is chosen much larger than the pulse duration (i.e., \( T_f \gg T_p \)), and \( T_s := N_f T_f \) the symbol duration. The effective bit rate is given by \( R_b := 1/T_s \). The length of the guard interval between subsequent pulses and subsequent symbols, \( T_g := T_f - T_p \), is typically chosen longer than the (expected) length of the UWB CIR, so as to circumvent IPI and ISI effects.

\[ h_i(t) = \sum_{\nu=0}^{L_C-1} \sum_{\mu=0}^{L_R-1} \lambda_{\mu,\nu}^{(i)} \delta(t - T_\nu^{(i)} - \tau_{\mu,\nu}^{(i)}), \quad (3) \]

where \( \lambda_{\mu,\nu}^{(i)} \) models the random multipath gain coefficient of the \( \mu \)th ray of the \( \nu \)th cluster, \( T_\nu^{(i)} \) the delay of the \( \nu \)th cluster, \( \tau_{\mu,\nu}^{(i)} \) the delay of the \( \mu \)th ray of the \( \nu \)th cluster, and \( \delta(\cdot) \)
denotes a Dirac impulse. The multipath gain coefficients are normalized such that \( \sum_{\nu=0}^{L_{\nu}-1} \sum_{\mu=0}^{L_{\mu}-1} (\lambda_{\mu,\nu})^2 = 1 \). In [27], four different parameter sets are specified for the various parameters in (3). The resulting channel models, CM1-CM4, represent different usage scenarios and entail different CIR lengths.

The channel gain is affected by log-normal fading and path loss, which is usually modeled in the UWB literature according to [28], [29]

\[
G(d) = G_0 + 10 \cdot \log(d_0/d) + \vartheta.
\]

Here, \( G(d) \) denotes the channel gain in dB, \( d \) the link length in meter, \( G_0 \) the channel gain resulting for some reference distance \( d_0 \) (e.g., \( d_0 = 1 \) m), \( p \) the path-loss exponent, and \( \vartheta \) the log-normal fading term. In the following, the source-destination (S-D) link will serve as a reference link for the path loss, in order to allow for a fair comparison between the proposed multi-hop relaying setup and the case of direct transmission. Assuming omni-directional antennas at all nodes, the relative channel gain associated with the \( i \)th link is thus modeled by a factor [23]

\[
A_i := \vartheta_i \left( \frac{d_{S-D}}{d_i} \right)^p,
\]

where \( \vartheta_i \) models the log-normal fading associated with Link \( i \), and \( d_{S-D} \) and \( d_i \) denote the lengths of the source-destination link and of Link \( i \), respectively. The path-loss exponent \( p \) is typically between \( 1.7 \leq p \leq 4.0 \) [27]. Throughout this paper, we assume that the lognormal shadowing terms \( \vartheta_i \) and \( \vartheta_i' \) associated with two different links \( i \neq i' \) are uncorrelated.

D. Proposed A&F Relay Structure

At the receiver front-end of each relay, the received signal is first passed through a bandpass filter \( h_{BP}(t) \) with one-sided bandwidth \( W \), so as to eliminate out-of-band noise (see Fig. 2 (b)). The filtered received signal at the \( i \)th relay, \( 1 \leq i \leq m - 1 \), is given by

\[
r_i(t) = \sqrt{A_i} \cdot h_i(t) \ast h_{BP}(t) \ast s_i(t) + n_i(t) = \sqrt{A_i} \cdot \alpha_i E_g \sum_{k=-\infty}^{\infty} \sum_{j=0}^{N_f-1} \xi_i[k] w_{rx,i}(t - jT_f - kT_s) + n_i(t),
\]

where \( A_i \) is the relative channel gain associated with Link \( i \), \( h_i(t) \) the corresponding CIR, \( s_i(t) \) the transmitted signal of the source node (\( i = 1 \)) or the \( (i-1) \)th relay (\( 1 < i \leq m - 1 \)), \( n_i(t) \) denotes filtered additive white Gaussian noise (AWGN) process with zero mean and single-sided noise PSD \( \frac{N_0}{2} \), \( \alpha_i \) is the transmit power allocation factor for the source node (\( i = 1 \)) or the \( (i-1) \)th relay (\( 1 < i \leq m - 1 \)), \( w_{rx,i}(t) := w_{rx}(t) \ast h_i(t) \ast h_{BP}(t) \) the received pulse associated with Link \( i \), and ‘\( \ast \)’ denotes linear convolution. Moreover,

\[
\xi_i[k] := \begin{cases} 
q_{m-i+2}[k] & \text{for } i = 1 \\
q_{m-i+2}[k] & \text{for } 1 < i \leq m - 1 \\
q_{m+1}[k] & \text{for } i = m - 1
\end{cases},
\]

where \( q_{m+1}[k] \) denotes the \( k \)th transmitted symbol of the source node and \( q_{m-i+2}[k] \) the (soft) estimate of the \( k \)th intermediate symbol \( q_{m-i+2}[k] \) formed at the \( (i-1) \)th relay (\( 1 < i \leq m - 1 \)), see details below. For the numerical results presented in Section V, the bandwidth \( W \) of the bandpass filter \( h_{BP}(t) \) was optimized numerically for maximization of the received SNR.

After the bandpass filter, (single) differential demodulation of the filtered receive signal \( r_i(t) \) is performed. To this end, signal \( r_i(t) \) is first delayed by a symbol duration \( T_s \) and is then multiplied by itself. The resulting signal \( r_i(t) r_i(t - T_s) \) is passed through an integrator with basic integration duration \( T_i \). Throughout this paper, we assume perfect synchronization of the relays and the destination node with respect to the employed frame structure [10]. Moreover, the overall integration time is composed of \( N_f \) separated sub-region intervals (one per received frame), as shown in Fig. 1 (b). By this means, the negative impact of the additive noise can be efficiently limited (e.g., in comparison to a single integration interval of length \( N_f T_f \)). The integrator at the \( i \)th relay yields the discrete-time output sample

\[
\tilde{q}_{m-i+1}[k] = \sum_{j=0}^{N_f-1} \int_{kT_s+jT_f}^{(k+1)T_s+jT_f} r_i(t) r_i(t - T_s) dt
\]

(cf. Fig. 2 (b)), which can be interpreted as a (soft) estimate of the intermediate symbol

\[
q_{m-i+1}[k] = q_{m-i+2}[k] \cdot q_{m-i+2}[k - 1]
\]

(due to the differential demodulation step, see Section III for further details). Similar to (2), the estimated symbols \( \tilde{q}_{m-i+1}[k] \) are finally modulated onto a signal

\[
s_{i+1}(t) = \sqrt{\alpha_{i+1} E_g} \sum_{k=-\infty}^{\infty} \sum_{j=0}^{N_f-1} \tilde{q}_{m-i+1}[k] w_{tx}(t - jT_f - kT_s),
\]

which is then re-transmitted to the next relay (\( 1 \leq i \leq m - 2 \)) or the destination node (\( i = m - 1 \)). Here, \( \alpha_{i+1} \) denotes the power allocation factor for the \( i \)th relay node.

E. Proposed Structure of the Destination Node

The receiver structure of the destination node is identical to that of the relays. In particular, we assume an identical bandpass filter \( h_{BP}(t) \) for simplicity. Similar to (6), the filtered received signal is given by

\[
r_m(t) = \sqrt{A_m} \cdot h_m(t) \ast h_{BP}(t) \ast s_m(t) + n_m(t) = \sqrt{A_m} \cdot \alpha_m E_g \sum_{k=-\infty}^{\infty} \sum_{j=0}^{N_f-1} \tilde{q}_2[k] w_{rx,m}(t - jT_f - kT_s) + n_m(t),
\]

where \( w_{rx,m}(t) := w_{tx}(t) \ast h_m(t) \ast h_{BP}(t) \) and \( h_m(t) \) denotes the CIR of the final link (Link \( m \)). In order to recover the symbols \( q_1[k] \) transmitted by the source node, the destination

\footnote{Similar to the filter bandwidth \( W \), the integration duration \( T_i \) has to be optimized such that (on average) most of the signal energy is captured, while the collected noise energy is kept to a minimum.}
node performs another differential demodulation step followed by an I&D operation. This yields estimated symbols
\[ \hat{q}_1[k] = \sum_{j=0}^{N_1} \int_{kT_s+jT_f+T_t}^{kT_s+(j+1)T_f+T_t} r_m(t) r_m(t-T_s) dt \] (12)
(c.f. Fig. 2 (c)), which are used for detection (uncoded transmission) or decoding (coded transmission) of the transmitted information bits. In the case of uncoded transmission, a simple slicer is used to quantize \( \hat{q}_1[k] \) to the detected symbols \( \hat{q}_1[k] \in \{\pm 1\} \). In the case of convolutionally encoded transmission, hard or soft input Viterbi decoding may be used to recover the information sequence [30]. Hard input Viterbi decoding has an inferior performance compared to soft input Viterbi decoding, cf. Section V, but also has a lower complexity, which may be a crucial advantage in high-speed UWB applications. For hard input Viterbi decoding, the estimated bits \( \hat{b}[n] = \{0, 1\} \) are obtained based on the hard symbol estimates \( \hat{q}_1[k] \). In contrast, for soft input Viterbi decoding, the Viterbi algorithm uses branch metrics of the form [30]
\[ M[k] = |\hat{q}_1[k] - \bar{\beta}_m \hat{q}_1[k]|^2, \] (13)
where \( \hat{q}_1[k] \in \{\pm 1\} \) is a trial symbol and \( \bar{\beta}_m \) is a gain factor which will be formally defined in the next section. We note that, since the noise component of \( \hat{q}_1[k] \) is non-Gaussian, cf. Section III, branch metric (13) is suboptimal. However, the actual noise distribution required to derive the optimal branch metric does not seem tractable. Moreover, typically the metric in (13) already yields high performance.

F. Comparison with Other Relaying Schemes
The proposed MD-A&F relaying scheme can be regarded as a generalization of narrowband A&F relaying to an UWB setting. As opposed to conventional narrowband A&F schemes, the relays in the proposed MD-A&F scheme include a demodulation stage, so as to account for the signal dispersion brought in by the wireless channel and, in particular, to collect the received signal energy at the relays. As the proposed MD-A&F scheme is tailored to a differential modulation framework, the demodulation stage at the relays consists of an autocorrelation of the received signal followed by an I&D operation. In addition to the demodulation stage, the relays also employ a corresponding re-modulation stage, so as to modulate the obtained soft estimates of the transmitted information or intermediate symbols onto the IR-UWB signal structure.

In Section V, we will compare the performance of the proposed MD-A&F scheme to that of alternative relaying schemes, particularly, two schemes that are based on D&F relaying, namely (i) D&F relaying with multiple-differential encoding at the source node (MD-D&F) and (ii) D&F relaying with single differential encoding at the source node (SD-D&F). Both relaying schemes employ the same demodulation — re-modulation relay structure as the proposed MD-A&F scheme. They thus represent corresponding wideband extensions of conventional narrowband D&F relaying schemes. However, in the MD-D&F scheme an additional slicer is employed at each relay, which is inserted between the I&D block and the pulse modulation block (cf. Fig. 2 (b)) to perform a hard decision on the transmitted symbols. The SD-D&F scheme, on the other hand, also requires an additional differential re-encoding block, which is inserted after the slicer, since differential encoding at the source node is conducted only once. The D&F-based schemes thus come at the expense of an increased relay complexity – even if no FEC decoding is employed at the relays.

Another aspect, which renders the considered MD-D&F and SD-D&F schemes costly, is the fact that – to the best of the authors’ knowledge – there are no analytical solutions for a (near-)optimal power allocation between the source node and the relay(s), neither for the dual-hop nor the multi-hop case. This is due to the lack of corresponding analytical expressions for the resulting effective SNR at the destination node. Correspondingly, for the simulation results of the D&F-based schemes presented in Section V, we have conducted brute-force searches for the optimal power allocation for each channel realization. This is quite complex and thus does not seem very practicable – especially not in the multi-hop case. As opposed to this, for the proposed MD-A&F scheme we derive a simple closed-form expression for the optimal power allocation in the dual-hop case and a simple suboptimal recursive power allocation solution for the multi-hop case, which permits a semi-distributed implementation with limited feedback between the nodes (for details see Section IV).

III. PERFORMANCE ANALYSIS
In this section, we provide a thorough performance analysis of the proposed IR-UWB MD-A&F relaying scheme. In the following, we show that the input-output behavior of the system between input symbol \( q_1[k] \) at the source node and the corresponding soft estimate \( \hat{q}_1[k] \) at the destination node can be described as
\[ \hat{q}_1[k] = \bar{\beta}_m q_1[k] + z_m[k], \] (14)
where \( \bar{\beta}_m \) and \( z_m[k] \) represent a gain factor and an effective noise sample with variance \( \sigma^2_m \), respectively. The additive noise \( z_m[k] \) is in general non-Gaussian. Its exact distribution does not seem tractable, and thus, an accurate analysis of the resulting bit error rate (BER) does not seem possible. Therefore, we focus on the effective SNR at the destination node in this section. In particular, we derive a closed-form expression for the effective SNR, \( \beta^2_m/\sigma^2_m \), for the dual hop case \( (m=2) \) and a recursive formula for the calculation of \( \beta^2_m \) and \( \sigma^2_m \) in the multi-hop case \( (m>2) \).

A. Dual-Hop Case
For the dual-hop case, double-differential encoding is performed at the source node. We start by substituting the filtered received signal \( r_1(t) \) at the relay, cf. (6) \((i=1)\), into (8) and get
\[ \hat{q}_2[k] = \bar{\beta}_1 q_3[k] q_3[k-1] + z_1[k], \] (15)

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3In Section V, we will also consider the performance of coherent A&F and D&F relaying schemes, where the demodulation stage (i.e., the autocorrelation and the subsequent I&D operation) is replaced by coherent S-Rake combining.
where
\[ \beta_1 := N_f A_1 \alpha_1 E_1, \quad E_1 := E_g \int_0^{T_1} w_{rx,1}(t) dt, \]
and
\[ z_1[k] := z'_{1,1}[k] + z'_{1,2}[k] + z'_{1,3}[k] \]
with \( z'_{1,1}[k], z'_{1,2}[k], \) and \( z'_{1,3}[k] \) being zero-mean noise terms defined as follows:
\[ z'_{1,1}[k] := q_3[k] \sqrt{A_1 \alpha_1 E_g} \sum_{j=0}^{N_f-1} \int_{kT_f+jT_1}^{kT_f+jT_1+t} n_1(t-t_x) dt, \]
\[ z'_{1,2}[k] := q_3[k-1] \sqrt{A_1 \alpha_1 E_g} \sum_{j=0}^{N_f-1} \int_{kT_f+jT_1}^{kT_f+jT_1+t} n_1(t) dt, \]
\[ z'_{1,3}[k] := \sum_{j=0}^{N_f-1} \int_{kT_f+jT_1}^{kT_f+jT_1+t} n_1(t) n_1(t-T_x) dt. \]
As described earlier, \( n_1(t) \) is obtained by filtering an AWGN process with single-sided noise PSD \( \frac{N_0}{2f_0} \) with a bandpass filter with one-sided bandwidth \( W \). The autocorrelation function \( \phi_1(\tau) \) of \( n_1(t) \) is thus given by [30]
\[ \phi_1(\tau) = E\{n_1(t)n_1(t-\tau)\} = \frac{N_0}{2} \sin(\pi W \tau) \pi W \cos(2\pi f_0 \tau), \]
where \( f_0 \) denotes the center frequency of the bandpass filter, and \( E\{\cdot\} \) denotes statistical expectation. Assuming that the bandwidth \( W \) is chosen sufficiently large, such that the frequency response of the received pulse \( w_{rx,1}(t) \) falls completely inside the PSD \( \Phi_1(f) \) of \( n_1(t) \), and the PSD \( \Phi_1(f) \) is sufficiently flat in the area of interest, the autocorrelation function \( \phi_1(\tau) \) can be replaced by \( \frac{N_0}{W} \delta(\tau) \). Thus, the variances of the noise terms \( z'_{1,1}[k], z'_{1,2}[k], \) and \( z'_{1,3}[k] \) can be approximated as
\[ \sigma^2_{z_{1,1}} = E\{z'_{1,1}^2[k]\} = N_f A_1 \alpha_1 E_g \int_0^{T_1} \int_0^{T_1} w_{rx,1}(t) dt \]
\[ \times \phi_1(\tau) \delta(\tau-t) d\tau \]
\[ \approx N_f A_1 \alpha_1 E_g \frac{N_0}{2} \int_0^{T_1} w_{rx,1}(t) dt = \frac{\beta_1 N_0}{2}, \]
\[ \sigma^2_{z_{1,2}} = E\{z'_{1,2}^2[k]\} = \sigma^2_{z_{1,1}}, \]
\[ \sigma^2_{z_{1,3}} = E\{z'_{1,3}^2[k]\} = N_f \int_0^{T_1} \int_0^{T_1} \phi_1^2(\tau) d\tau dt \]
\[ \approx N_f \int_0^{T_1} \int_{T_1-t}^{T_1} \phi_1^2(u) du dt \]
\[ = N_f \int_0^{T_1} \frac{N_0^2}{4} \cdot W dt = \frac{W N_f T_1 N_0^2}{2}. \]
In (22), we have exploited the fact that the integrand \( \phi_1^2(u) \) is Dirac-like, i.e., the integral vanishes outside \([-t, T_1-t]\), and have employed Parseval’s theorem to calculate the integral. Altogether, the variance of the noise term \( z_1[k] = z'_{1,1}[k] + z'_{1,2} + z'_{1,3}[k] \) in (15) is given by
\[ \sigma^2_{z_1} := \beta_1 N_0 + W N_f T_1 N_0^2/2. \]
Note that \( z_1[k] \) is not Gaussian distributed, since \( z'_{1,3}[k] \) is not Gaussian.

Along the same lines, the integrator output of the destination node can be analyzed. Substituting (11) into (12) \((m = 2)\) yields
\[ \tilde{q}_1[k] = \beta_2 \tilde{q}_2[k] q_2[k-1] + z_2'[k], \]
where \( \beta_2 := N_f A_2 \alpha_2 E_2, \quad E_2 := E_g \int_0^{T_1} w_{rx,2}(t) dt, \)
and the noise term \( z_2'[k] = z'_{2,1}[k] + z'_{2,2}[k] + z'_{2,3}[k] \) is similarly defined as \( z_1[k] \) in (15)-(18), with \( q_2[k] \) being replaced by \( \tilde{q}_2[k] \). \( A_1 \) by \( A_2, \alpha_1 \) by \( \alpha_2, n_1(t) \) by \( n_2(t) \), and \( w_{rx,1}(t) \) by \( w_{rx,2}(t) \).
Plugging \( \tilde{q}_2[k] \) from (15), into (23) we get
\[ \tilde{q}_1[k] = \beta_2 \tilde{q}_2[k] q_2[k-1] + z_2'[k] + z'_{2,4}[k] + z'_{2,5}[k] + z'_{2,6}[k], \]
where \( z'_{2,4}[k], z'_{2,5}[k], \) and \( z'_{2,6}[k] \) are given by
\[ z'_{2,4}[k] = \beta_2 \tilde{q}_2[k] z_1[k-1], \]
\[ z'_{2,5}[k] = \beta_2 \beta_1 q_2[k-1] z_1[k], \]
\[ z'_{2,6}[k] = \beta_2 z_1[k] z_1[k-1]. \]
The variances of \( z'_{2,4}[k], z'_{2,5}[k], \) and \( z'_{2,6}[k] \) can be calculated similarly to (20)-(22). One obtains
\[ \sigma^2_{z_{2,4}} = \sigma^2_{z_{2,5}} = \sigma^2_{z_{2,6}} = \frac{\beta_2^2 \sigma^2_2}{2}, \]
\[ \sigma^2_{z_{2,4}} = \sigma^2_{z_{2,5}} = \sigma^2_{z_{2,6}} = \frac{\beta_2^2 \sigma^2_4}{2}, \]
\[ \sigma^2_{z_{2,4}} = \sigma^2_{z_{2,5}} = \sigma^2_{z_{2,6}} = \frac{\beta_2^2 \sigma^2_4}{2}. \]
For computing the variance of \( z'_{2,6}[k] \), we have assumed that \( z_1[k] \) and \( z_1[k-1] \) are uncorrelated. Based on (24)-(29), we are now ready to obtain the input-output behavior of our system according to (14) for \( m = 2 \), where \( \beta_2 := \beta_2 \beta_1^2 \) and \( z_2[k] := z'_{2,4}[k] + z'_{2,5}[k] + z'_{2,6}[k]. \) In order to calculate the variance of \( z_2[k] \), denoted as \( \sigma^2_2 \) in the sequel, we should note that \( z'_{2,4}[k] \) and \( z'_{2,5}[k] \) are not mutually independent, i.e., their correlation cannot be ignored. In particular, one finds that
\[ E\{z'_{2,4}[k] z'_{2,5}[k]\} = (\beta_2 \beta_1)^2 E\{q_2[k] q_2[k-1] z_1[k] z_1[k-1]\} \]
\[ = (\beta_2 \beta_1)^2 E\{q_2[k] q_2[k-1] z_1'[k] z_1'[k-1]\}. \]
A careful look at (16) and (17) reveals that
\[ q_2[k-2] z'_{1,2}[k-1] = q_3[k] z'_{1,1}[k]. \]
Thus, we have
\[ E\{q_2[k] q_2[k-1] z'_{1,2}[k-1]\} = E\{q_3[k] q_3[k-2] z'_{1,1}[k] z'_{1,1}[k-1]\} \]
\[ = E\{z'_{1,1}[k]\} = \beta_1 N_0 \]
and get
\[ E\{z'_{2,4}[k] z'_{2,5}[k]\} = \beta_2^2 \beta_1^2 N_0 \].
Based on (14) and (24)-(31), the effective SNR at the destination node can be calculated as

\[
\text{SNR} = \frac{\beta^2}{\sigma^2} = \frac{(\beta_2 \beta_1)^2}{\sigma_{z_2}^2 + \sigma_{z_5}^2 + \sigma_{z_6}^2 + 2E\{z_{2,A}[k]z_{5,A}[k]\} + \sigma_{z_2}^2} \approx \frac{(\beta_2 \beta_1)^2}{\varphi_2 + \psi_2 + \frac{WN_T N_2^2}{2}}.
\]  

where \(\varphi_2 := \beta_2^2 (2 \beta_1^2 \sigma_1^2 + \beta_1^4 N_0 + \sigma_1^4)\) and \(\psi_2 := \beta_2 N_0 (\beta_1^2 + \sigma_1^2)\). Eq. (32) will be exploited in Section IV to find the optimal power allocation for the source node and the relay.

**B. Multi-Hop Case**

In the following, we turn to the case of multiple hops \((m > 2)\). In order to obtain an expression for the gain factor \(\tilde{\beta}_m\), the noise term \(z_m[k]\), and the corresponding noise variance \(\sigma_m^2\) (and finally the effective SNR \(\beta_m^2/\sigma_m^2\) at the destination node), the following recursion steps are required (similar to the derivation for the dual-hop case in Section III-A):

- Assume that we have already obtained the input-output relation for the soft estimate \(\tilde{q}_{m-i+1}[k]\) formed by the \(i\)th relay, according to

\[
\tilde{q}_{m-i+1}[k] = \beta_i q_{m-i+1}[k] + z_i[k] \quad (i = 1) \text{ or } \tilde{q}_{m-i+1}[k] = \tilde{\beta}_i q_{m-i+1}[k] + z_i[k] \quad (1 < i < m),
\]

(cf. (14) and (15) for the dual-hop case \((m = 2)\). In particular, assume that we have calculated the gain factor \(\beta_i\) or \(\tilde{\beta}_i\) and the variance \(\sigma_i^2\) of the noise term \(z_i[k]\). In the sequel, we focus on the case \(i > 1\) for simplicity. For \(i = 1\), parameter \(\beta_i\) needs to be replaced by \(\tilde{\beta}_i\) in the following equations.

- For the integrator output of the \((i+1)\)th relay \((i < m-1)\) or the destination node \((i = m-1)\), we obtain the relation

\[
\tilde{q}_{m-i}[k] = \beta_{i+1} \tilde{q}_{m-i+1}[k] \tilde{q}_{m-i+1}[k] \tilde{q}_{m-i+1}[k] + z_{i+1}[k],
\]

where

\[
\beta_{i+1} := N_f A_{i+1} \alpha_{i+1} E_{i+1},
\]

\[
E_{i+1} := E_g \int_0^{T_i} w_{r_{x,i+1}}(t)dt,
\]

and \(z_{i+1}[k]\) represents a zero-mean noise term with components defined similar to (16)-(18), with \(q_{3}[k]\) being replaced by \(q_{m-i+1}[k]\), \(A_1\) by \(A_{i+1}\), \(\alpha_1\) by \(\alpha_{i+1}\), \(n_{i+1}(t)\) by \(n_{i+1}(t)\), and \(w_{r_{x,1}}(t)\) by \(w_{r_{x,i+1}}(t)\). The variance of the noise term \(z_{i+1}[k]\) is calculated as

\[
\sigma_{z_{i+1}}^2 \approx \frac{(\beta_1^2 + \sigma_1^2)\beta_{i+1} N_0 + WN_j T_i N_0^2}{2}.
\]

- Plugging the expression for \(\tilde{q}_{m-i+1}[k]\) from (34) into (35), we obtain a new representation for the soft estimate \(\tilde{q}_{m-i}[k]\), according to

\[
\tilde{q}_{m-i}[k] = \beta_{i+1} q_{m-i}[k] + z_{i+1}[k].
\]

The gain factor \(\tilde{\beta}_{i+1}\) is obtained via the recursion

\[
\tilde{\beta}_{i+1} = \beta_{i+1} \beta_i^2
\]

with initialization \(\tilde{\beta}_0 := 1\). Moreover, making similar approximations as in Section III-A and taking the correlations between the involved noise terms into account, the variance \(\sigma_{i+1}^2\) of the noise term \(z_{i+1}[k]\) can be calculated via the recursion

\[
\sigma_{i+1}^2 \approx \tilde{\beta}_{i+1} \left(2 \tilde{\beta}_1^2 (\sigma_i^2 + \eta_i) + \sigma_i^4\right) + \beta_{i+1} N_0 (\tilde{\beta}_{i+1}^2 + \sigma_i^2) + WN_j T_i N_0^2 / 2,
\]

where

\[
\eta_i \approx \tilde{\beta}_1^2 \beta_{i-1}^2 (\sigma_{i-1}^2 + 2 \eta_{i-1}) + \beta_{i-1} \tilde{\beta}_{i-1} \beta_{i-1} N_0 / 2.
\]

At the end of the recursion \((i = m-1)\), we thus obtain the following expression for the effective SNR at the destination node:

\[
\text{SNR} = \frac{\tilde{\beta}_m^2}{\sigma_m^2} \approx \frac{(\beta_m \tilde{\beta}_{m-1})^2}{\varphi_m + \psi_m + \frac{WN_T N_2^2}{2}},
\]

where \(\varphi_m := \beta_m^2 (2 \tilde{\beta}_{m-1}^2 (\sigma_{m-1}^2 + \eta_{m-1}) + \sigma_{m-1}^4)\) and \(\psi_m := \beta_m N_0 (\tilde{\beta}_{m-1}^2 + \sigma_{m-1}^2)\).

**IV. Optimized Transmit Power Allocation**

Based on the effective SNR results presented in the previous section, we next conduct an analytical optimization of the transmit power allocation factors \(\alpha_i\) for the source node \((i = 1)\) and the relay nodes \((2 < i < m-1)\). Similar to Section III, we focus first on the dual-hop case, before we consider the more challenging multi-hop case. We also show that the proposed suboptimal power allocation algorithm for the multi-hop case can be implemented in a semi-distributed manner with limited message exchange between nodes.

Note that existing power allocation solutions for narrowband A&F relaying schemes cannot be readily adopted for the MD-A&F scheme. This is due to the rather specific relay structure under consideration, which is tailored to the IR-UWB signal structure and a differential modulation framework. Also, note that similar results as presented in the following are not available for the MD-D&F or the SD-D&F scheme (cf. Section II-F), as analytical expressions for the resulting effective SNR at the destination node do not seem tractable.

**A. Dual-Hop Case**

Based on (32), we can optimize the transmit power allocation factors \(\alpha_1\) and \(\alpha_2\) for the source node and the A&F relay,
respectively. We aim to maximize the effective SNR at the destination node, under the constraint of keeping the total transmit power per symbol period, $N_f E_g$, fixed.

It can be shown that the total transmit power constraint can be expressed as

$$N_f E_g \alpha_1 + N_f E_g \alpha_2 E_{rx,1} = N_f E_g,$$  \hspace{1cm} (41)

where $E_{rx,1} := \beta_1^2 + \sigma_1^2$ is the power of the received signal at relay $R_1$. Concerning (32), we can make a high-SNR approximation, according to

$$\text{SNR} \approx \frac{\beta_2 \beta_1^2}{\beta_2 (2 \sigma_1^2 + \beta_1 N_0) + N_0},$$  \hspace{1cm} (42)

where we have used that, for high SNR values, $2 \beta_2^2 \sigma_1^2 + \beta_1^2 N_0 \gg \sigma_1^2, \beta_2^2 \gg \sigma_1^2$, and $WNF_1 F_1 N_0^2 / 2$ becomes negligible.

Assuming again high SNR, we can now formulate the Lagrange problem

$$\Lambda(\alpha_1, \alpha_2, \gamma) = \frac{\beta_2 \beta_1^2}{(3 \beta_1 \beta_2 + 1) N_0} - \gamma (\alpha_1 + \alpha_2 \beta_1^2 - 1),$$  \hspace{1cm} (43)

where $\gamma$ denotes the Lagrange multiplier. In order to arrive at (43), we have used the high-SNR approximations $\beta_1^2 \gg \sigma_1^2$ and $\sigma_1^2 \approx \beta_1 N_0$. Based on (43), we obtain the optimal transmit power factors as

$$\alpha_1^* = \frac{1}{1 + \sqrt{\frac{A_1 E_1}{3 A_2 E_2}}} = \frac{1}{1 + \sqrt{\frac{\sigma_1}{\sigma_2} \left( \frac{E}{m} \right)^2}},$$

$$\alpha_2^* = \frac{1 - \alpha_1^*}{E_{rx,1}},$$  \hspace{1cm} (44)

which only depend on the effective link gains $A_1 E_1$ and $A_2 E_2$.

The numerical results in Section V will reveal that (44) is, in fact, very close to the optimal solution (also for moderate SNR values). For the computation of the power allocation factors in (44), the source and the relay have to have access to estimates of the effective link gains $A_1 E_1$ and $A_2 E_2$, and the relay also has to estimate the power of its received signal $E_{rx,1}$. From (44), we can see that if $d_1 \ll d_2$ holds, i.e., the relay is very close to the source node, the power allocation factor for the source node becomes very small. This is reasonable, since in order to achieve a high effective SNR at the destination node most of the available transmit power should be allocated to the relay, so as to bridge the long link from the relay to the destination. Vice versa, if the relay is very close to the destination node, most of the available transmit power is allocated to the source node. Thus, by (44) the individual SNRs for the two hops are balanced such that an optimal end-to-end SNR is achieved. The same idea will also be employed in the following subsection, in order to construct a (suboptimal) power allocation solution for the multi-hop case.

B. Multi-Hop Case

A closed-form solution for the optimal transmit power allocation factors $\alpha_i$ ($1 \leq i \leq m$) does not seem feasible in the multi-hop case ($m > 2$) because of the involved recursive expression (40) for the SNR. Thus, in the following, we present a heuristic algorithm for finding near-optimal values for the transmit power allocation factors $\alpha_i$. The proposed algorithm is based on the optimization presented in Section IV-A for the dual-hop case. Essentially, we decompose the $m$-hop transmission into $m-1$ two-hop transmissions, cf. Fig. 3 (a).

In particular, we first assume that there are only three nodes, namely the source node (Node 1), the last relay (Node $m-1$), and the destination node (Node $m+1$) and find the corresponding transmit power factor for the source-relay link based on (44). In the next step, the transmit power allocated to the source node in the previous step is regarded as the overall available transmit power, and the power allocation between the source node and the $(m-2)$th relay $R_{m-2}$ (Node $m-1$) is calculated assuming that the $(m-1)$th relay $R_{m-1}$ (Node $m$) is the destination. We repeat this method until we find the overall power allocation for the source node and subsequently calculate the powers allocated to each relay in a similar manner.

In the following, let $A_{i,j}$ denote the gain factor associated with the link from Node $i$ to Node $j$, cf. (5), where $i = 1$ represents the source node and $j = m+1$ the destination node. For the corresponding energies of the received pulses, $E_{i,j}$, we assume for simplicity that $E_{i,j} = 1$ for all indices $i, j$. Assuming that we have only the source (Node 1), the last relay $R_{m-1}$ (Node $m$), and the destination (Node $m+1$), according to (44), the power factor for the source, $\zeta_{m-1}^1$, is given by

$$\zeta_{m-1}^1 = \frac{1}{1 + \sqrt{\frac{A_1}{3 A_{m,m+1}}}}.$$  \hspace{1cm} (45)

The normalized fraction of power allocated to the $(m-1)$th relay $R_{m-1}$ (Node $m$) is given by $1 - \zeta_{m-1}^1$. However, we cannot calculate the associated power allocation factor $\alpha_m$, since the received power of the relay is not yet known. In the next step we take into account that the received signal at the $(m-1)$th relay $R_{m-1}$ (Node $m$) actually originates from the $(m-2)$th relay $R_{m-2}$ (Node $m-1$). To find the corresponding power allocation, we consider only the source, relay $R_{m-2}$, and relay $R_{m-1}$, with the latter playing the role of the destination. Therefore, the fraction of power allocated to the source is now given by $\zeta_{m-2}^1 \zeta_{m-1}^1$, where $\zeta_{m-1}^1 := 1 / (1 + \sqrt{A_{m-1,m-1} / 3 A_{m-2,m-1}})$. This procedure is repeated until we arrive at the first relay $R_1$ (Node $i = 2$), which leads to the following final power allocation factor for the source:

$$\alpha_1^* = \prod_{i=2}^{m} \zeta_i^1,$$  \hspace{1cm} (46)

where $\zeta_i^1$ is defined as in (45) (with $m = i$). Knowing the power allocated to the source, the available power for allocation to the $m-1$ relays is given by $(1 - \alpha_1^*) N_f E_g$. The power allocation factor for relay $R_1$ is constrained by

$$\alpha_2 E_{rx,2} + E_2 = 1 - \alpha_1^*,$$  \hspace{1cm} (47)
Fig. 3. (a) Illustration of the optimization of $\alpha_1$ using a recursive approach; (b) Illustration of the optimization of $\alpha_2$ for known $\alpha_1^*$. The power allocation factor $\alpha_2$ of the first relay $R_1$ (Node $i = 2$) is now obtained using the same recursive approach as for the source node before, i.e., the first relay $R_1$ now plays the role of the source node, see Fig. 3 (b). Subsequently, the same steps are repeated for relays $R_2$ to $R_{m-1}$. For the $j$th relay $R_j$, $1 \leq j \leq m-1$, this leads to the power allocation factor

$$\alpha_j^* = \frac{\kappa_j}{E_{rx,j}} \prod_{i=j+2}^m \zeta_i^{j+1},$$

with

$$\zeta_i^j = \frac{1}{1 + \sqrt{\frac{A_{i,j}}{3A_{i+1,j+1}}}},$$

$$\kappa_j = \kappa_{j-1} - \alpha_j^* E_{rx,j-1},$$

$$E_{rx,j} = (A_j \alpha_j E_{rx,j-1} N_j E_g)^2 + \sigma_j^2,$$

where $E_{rx,1} := \beta_1^2 + \sigma_1^2$ denotes the received power of the first relay $R_1$ and $E_2$ is the remaining power to be allocated to the other $m - 2$ relays later on. The power allocation factor $\alpha_2$ of the first relay $R_1$ (Node $i = 2$) is now obtained using the same recursive approach as for the source node before, i.e., the first relay $R_1$ now plays the role of the source node, see Fig. 3 (b). Subsequently, the same steps are repeated for relays $R_2$ to $R_{m-1}$. For the $j$th relay $R_j$, $1 \leq j \leq m-1$, this leads to the power allocation factor

$$\alpha_j^* = \frac{\kappa_j}{E_{rx,j}} \prod_{i=j+2}^m \zeta_i^{j+1},$$

with

$$\zeta_i^j = \frac{1}{1 + \sqrt{\frac{A_{i,j}}{3A_{i+1,j+1}}}},$$

$$\kappa_j = \kappa_{j-1} - \alpha_j^* E_{rx,j-1},$$

$$E_{rx,j} = (A_j \alpha_j E_{rx,j-1} N_j E_g)^2 + \sigma_j^2,$$

where the received power at the $j$th relay $R_j$, $E_{rx,j}$, and normalization factor $\kappa_j$ are calculated recursively for $j \geq 1$. For initialization, we have $\kappa_0 := 1$ and $E_{rx,0} := 1$. It is easy to check that the suboptimal power allocation in (48) fulfills the overall power constraint

$$N_f E_g \alpha_1^* + \sum_{i=2}^m \alpha_i^* N_f E_{rx,i-1} = N_f E_g.$$

Although the proposed heuristic power allocation scheme for the multi-hop case is suboptimal, our results in the next section confirm its near-optimal performance. Furthermore, the proposed power allocation algorithm allows for a semi-distributed implementation, where the source has to have only access to estimates of its own path loss to all relays and the path-losses between all neighboring relays. In particular, for the proposed semi-distributed implementation, the source computes the factors $f_j := \prod_{i=j+1}^m \zeta_i^j$, $1 \leq j \leq m$, feeds back $f_{j+1}$ to relay $R_j$, computes $\alpha_j^* = f_j$, and feeds back $\kappa_{j+1}$ to relay $R_{j+1}$. Relay $R_j$, $1 \leq j \leq m-2$, estimates its own received power $E_{rx,j}$, computes $\alpha_j^* + 1$ based on (48), computes $\kappa_{j+1}$ based on (50), and feeds back $\kappa_{j+1}$ to the next relay $R_{j+1}$. The last relay $R_{m-1}$ only has to estimate $E_{rx,m-1}$ and compute $\alpha_{m}^*$. We note that computation of the optimal power allocation based on (40) requires a centralized approach where the central node (e.g. the source or the destination node) has to perform an exhaustive search over a fine-scaled grid of all possible power allocation factors, since (40) is not convex in the power allocation factors. Such an exhaustive search is computationally expensive for more than two hops.
Fig. 4. Effective SNR at the destination node versus transmit power allocation factor of the source node $\alpha_1$ for dual-hop transmission (three different values for the source-relay link length are considered).

Fig. 5. BER at the destination versus $E_b/N_0$ for double differential, single differential, and coherent uncoded transmission over two hops with both A&F and D&F relaying (geometrical setting with $\rho=0.2$).

V. NUMERICAL PERFORMANCE RESULTS

In the following, we present numerical performance results, which illustrate the excellent performance of our proposed MD-A&F scheme and corroborate our analysis in Sections III and IV. We start with considering the dual-hop case without additional channel coding. Afterwards, we will consider the benefits of an outer FEC scheme and present results for the multi-hop case.

In the following, the information symbols $q_1[k] \in \{\pm 1\}$ are transmitted in blocks of 1000 symbols. The CIRs are assumed to remain static for the duration of an entire block (70 $\mu$s). As an example, we focus on channel model CM1 in the sequel and assume a path-loss exponent of $p=3$, unless specified otherwise. One frame is used for transmitting a single information symbol ($N_f=1$), and the frame length is chosen such that the guard interval between subsequent pulses is larger than the root-mean-square (rms) delay spread of the channel ($T_f=70$ ns), so as to circumvent ISI effects. For the transmitted pulse $w_{tx}(t)$, we employ the widely-used second derivative of a Gaussian pulse [4], i.e.,

$$w_{tx}(t) = [1 - 4\pi(t-v_p)/v_m^2] \exp[\frac{-2\pi((t-v_p)/v_m)^2}],$$

where $v_p=0.35$ ns and $v_m=0.2877$ ns ($T_p=0.7$ ns). The bandwidth $W$ of the bandpass filter is optimized such that for a single link the maximum received SNR is obtained ($W=5$ GHz). Similarly, the integration time $T_i$ has been optimized such that on average the maximum effective SNR at the integrator output is obtained ($T_i=5.25$ ns).

Fig. 4 shows the effective SNR at the destination node for the MD-A&F relaying scheme for the dual-hop case and $E_b/N_0=9$ dB, considering three different positions of the relay ($\rho := d_t/d_{s-D} = \{0.2,0.4,0.8\}$), where $d_t$ and $d_{s-D}$ denote the lengths of the source-relay and the source-destination link, respectively. Analytical results based on (32) are represented by lines, and corresponding simulation results are represented by markers. Moreover, the cross signs indicate the (near-)optimum value for the transmit power allocation factor $\alpha_1$ for the source node, which was found based on (44). As can be seen, the analytical results and the simulation results are in good agreement, and considering the fact that our (approximate) formula (44) was derived for high SNR values while the SNR considered in Fig. 4 is rather moderate, the power allocation factor $\alpha_1^{\ast}$ obtained with (44) offers a remarkable accuracy.

Fig. 5 illustrates the performance of the proposed non-coherent MD-A&F relaying scheme with double-differential encoding at the source node, obtained by means of Monte-Carlo simulation using a large number of independent CIR realizations. We have considered a geometrical setting, where the relay is located relatively close to the source node ($\rho = 0.2$). The proposed MD-A&F scheme is compared with (i) direct transmission from the source to the destination node, (ii) non-coherent A&F relaying with single differential encoding at the source node (SD-A&F), (iii) non-coherent D&F relaying with double-differential encoding at the source node (MD-D&F), (iv) non-coherent D&F relaying with single-differential encoding at the source node (SD-D&F), and (v) coherent A&F and D&F relaying with S-Rake combining at the relay and at the destination node using $L=3,5$ Rake fingers, cf. Section II-F. For the proposed MD-A&F scheme, the transmit power allocation factors $\alpha_1$ and $\alpha_2$ for the source node and the relay, respectively, were optimized based on (44). For the coherent A&F relaying scheme with S-Rake combining at the relay and at the destination node, we have used a similar closed-form power allocation solution as for the proposed MD-A&F scheme (details have been omitted here for the sake of conciseness). For all other schemes, especially the D&F-based schemes, we had to resort to simulations (exhaustive search), due to the lack of analytical power allocation solutions.

As can be seen, all considered schemes offer substantial
performance improvements over direct transmission, which is due to significant path-loss gains. These performance improvements can, for example, be translated into an extended coverage [23]. For sufficiently high SNR values, our proposed MD-A&F scheme outperforms the coherent A&F and D&F relaying schemes, unless a relatively large number of Rake fingers is employed (e.g., \( L \geq 5 \)), and it also outperforms the SD-D&F scheme. The MD-D&F scheme offers a small performance advantage of about 0.5 dB compared to the MD-A&F scheme (at a BER of \( 10^{-4} \)), at the expense of an increased relay complexity. The SD-A&F relaying scheme suffers from a significant loss in performance compared to the proposed MD-A&F scheme, due to excessive ISI. We have doubled the integration time \( T_i \) at the receiver for SD-A&F relaying, since the effective CIR seen at the destination node is the convolution of the CIRs of the source-relay and the relay-destination links. We note that the performance of the SD-A&F scheme could be improved by increasing the frame duration \( T_f \) at the expense of a loss in data rate.

Next, we consider the performance of the proposed MD-A&F relaying scheme, when an outer FEC scheme is used, cf. Fig. 2. The FEC block is placed before the double-differential encoder and includes a convolutional encoder and a bit interleaver. In particular, we adopted the quasi-standard rate-1/2 binary convolutional code with generator polynomials \([133,171]_8\) [30]. The performance of the proposed MD-A&F relaying scheme as well as that of the considered alternative coherent and non-coherent relaying schemes is shown for hard and soft input Viterbi decoding in Figs. 6 and 7, respectively. A comparison of Figs. 5 and 6 shows that FEC with hard input Viterbi decoding improves the performance significantly (compared to uncoded transmission). For example, for MD-A&F relaying the coded scheme yields a performance gain of about 3.8 dB compared to the uncoded scheme at a BER of \( 10^{-4} \). Note that this gain comes at the expense of a loss in data rate and an increase in receiver complexity (destination node).

As can be seen, the performance of the proposed MD-A&F scheme is now comparable to that of the MD-D&F scheme and is still superior to that of the SD-D&F and SD-A&F schemes. We have also included simulation results for the case of equal power allocation (dotted lines). As can be seen, all considered schemes – both D&F-based and A&F-based – suffer from notable performance degradations compared to the case with optimal power allocation. In other words, an optimization of the power allocation factors \( \alpha_1 \) and \( \alpha_2 \) is highly desirable. Here, the advantage of our proposed MD-A&F scheme over the D&F-based schemes becomes evident, since for the MD-A&F scheme the simple closed-form power allocation solution (44) can be employed instead of an exhaustive search.

Fig. 7 reveals that for the proposed MD-A&F relaying scheme additional performance gains are possible with soft input Viterbi decoding (about 1.2 dB compared to hard input Viterbi decoding at a BER of \( 10^{-4} \)). Contrary to this, for the MD-D&F scheme soft input Viterbi decoding does not result in a notable gain, and there is a performance gap of about 1.2 dB and 1 dB compared to the proposed MD-A&F scheme and the SD-D&F scheme, respectively, at a BER of \( 10^{-4} \). Altogether, it seems that the performance of the D&F-based schemes is somewhat limited by the decision errors at the relays. In particular, the simple Euclidean distance metric (13) used for soft input Viterbi decoding is highly suboptimal in this case since the reliability information (i.e., the amplitude of \( \tilde{q}_k[k] \)) exploited for decoding is compromised by the decision errors. This problem does neither exist for the D&F-based schemes with hard input Viterbi decoding, where reliability information is not exploited, nor for the proposed MD-A&F...

\[ \text{BER at the destination versus } E_b/N_0 \text{ for double differential, single differential, and coherent coded transmission over two hops with both A&F and D&F relaying, using hard input Viterbi decoding (geometrical setting with } \rho = 0.2). \text{ Dotted lines indicate the performance with equal power allocation.} \]

\[ \text{BER at the destination versus } E_b/N_0 \text{ for double differential, single differential, and coherent coded transmission over two hops with both A&F and D&F relaying, using soft input Viterbi decoding (geometrical setting with } \rho = 0.2). \]

\[ \text{As can be seen, the performance of the proposed MD-A&F scheme is now comparable to that of the MD-D&F scheme and is still superior to that of the SD-D&F and SD-A&F schemes.}\]

\[ \text{We have also included simulation results for the case of equal power allocation (dotted lines). As can be seen, all considered schemes – both D&F-based and A&F-based – suffer from notable performance degradations compared to the case with optimal power allocation. In other words, an optimization of the power allocation factors } \alpha_1 \text{ and } \alpha_2 \text{ is highly desirable. Here, the advantage of our proposed MD-A&F scheme over the D&F-based schemes becomes evident, since for the MD-A&F scheme the simple closed-form power allocation solution (44) can be employed instead of an exhaustive search.}\]

\[ \text{Fig. 7 reveals that for the proposed MD-A&F relaying scheme additional performance gains are possible with soft input Viterbi decoding (about 1.2 dB compared to hard input Viterbi decoding at a BER of } 10^{-4}). \text{ Contrary to this, for the MD-D&F scheme soft input Viterbi decoding does not result in a notable gain, and there is a performance gap of about 1.2 dB and 1 dB compared to the proposed MD-A&F scheme and the SD-D&F scheme, respectively, at a BER of } 10^{-4}. \text{ Altogether, it seems that the performance of the D&F-based schemes is somewhat limited by the decision errors at the relays. In particular, the simple Euclidean distance metric (13) used for soft input Viterbi decoding is highly suboptimal in this case since the reliability information (i.e., the amplitude of } \tilde{q}_k[k] \text{) exploited for decoding is compromised by the decision errors. This problem does neither exist for the D&F-based schemes with hard input Viterbi decoding, where reliability information is not exploited, nor for the proposed MD-A&F...}\]
scheme with soft input Viterbi decoding, where no preliminary decisions are made at the relays.

Fig. 8 shows further performance results for the case of soft input Viterbi decoding, this time for a geometrical setting where the relay is located half-way in between the source node and the destination node ($\rho = 0.5$). As can be seen, in this case the proposed MD-A&F scheme has an inferior performance compared to the D&F-based schemes (the performance difference at a BER of $10^{-4}$ is about 1 dB and 0.5 dB, respectively). However, note again that the excellent performance of the D&F-based schemes shown in Fig. 8 is based on an exhaustive search for the optimal power allocation factors $\alpha_1$ and $\alpha_2$, whereas for the proposed MD-A&F scheme the simple closed-form solution (44) can be utilized. A major advantage of our proposed MD-A&F scheme is also that even in the multihop case we have derived a (suboptimal) solution for the power allocation factors $\alpha_1$ (cf. Section IV-B), whereas an exhaustive search would soon become prohibitive when the number of hops increases.

We now turn our attention to the multi-hop case. Fig. 9 shows the effective SNR at the destination node versus $E_s/N_0$ for three-hop ($m = 3$) and four-hop ($m = 4$) transmission with MD-A&F relaying. The distances between the network nodes normalized by the source-destination distance are given by $0.5 – 0.4 – 0.1$ (3-hop case 1), $0.6 – 0.3 – 0.1$ (3-hop case 2), $0.25 – 0.25 – 0.25$ (4-hop case 1), and $0.1 – 0.3 – 0.2 – 0.4$ (4-hop case 2), respectively. The analytical results (solid lines) were obtained with the recursive method for calculation of the effective SNR outlined in Section III-B. The simulation results (markers) in Fig. 9 confirm the accuracy of the derived analytical SNR expression (40) especially for moderate to high SNR. For low SNR, the approximations made to arrive at (40), e.g., concerning the assumption that certain noise terms are uncorrelated, are less justified, so that the analytical results slightly deviate from the simulations. Finally, we investigate the effectiveness of the recursive power allocation method proposed in Section IV-B for the multi-hop case. In particular, we show in Tables I and II the power allocation factors obtained with the proposed method and with an exhaustive search over a fine grid of all possible power allocation factors for 3-hop and 4-hop transmission, respectively. We also show the resulting effective SNRs at the destination node, both for the proposed power allocation and for equal power allocation to all nodes. We chose $E_s/N_0 = 12.8$ dB for both tables, and the path-loss exponents were set to $p = 3$ and $p = 4$ for the 3-hop and the 4-hop case, respectively. Tables I and II confirm the near-optimality of the proposed recursive power allocation method for all considered relay arrangements. In particular, large performance gains are achieved compared to the case of equal power allocation.

VI. CONCLUSION

We proposed a multiple-differential encoding scheme for multi-hop A&F relaying in IR-UWB systems. In contrast to conventional A&F relaying with single-differential encoding – as typically suggested for non-coherent narrow-band systems – the proposed scheme overcomes the UWB-specific problem of ISI accumulation by performing multiple-differential encoding at the source node and single-differential demodulation at each relay and the destination node. We have provided a closed-form expression for the effective SNR at the destination node for dual-hop transmission and a recursive method for SNR calculation for multi-hop transmissions. Furthermore, we derived a closed-form expression for the optimal power allocation for the dual-hop case and a near-optimal recursive method for power allocation in the multi-hop case. Simulation results for
uncoded and coded transmission showed that the proposed multiple-differential A&F-relaying scheme offers an excellent performance. In particular, despite its low complexity it can compete with non-coherent D&F-based schemes and even with coherent A&F- and D&F-relaying schemes that are based on S-Rake combining. In particular, the proposed transmit power allocation solutions were shown to offer significant performance improvements over equal power allocation to all nodes.

REFERENCES


### TABLE I

<table>
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<tr>
<th>Relative position of nodes</th>
<th>Type of optimization</th>
<th>$\alpha_1^*$</th>
<th>$\alpha_2^*E_{r,x,1}$</th>
<th>$\alpha_3^*E_{r,x,2}$</th>
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Jan Mietzner (S’02-M’08) was born in Rendsburg, Germany, in 1975. He studied electrical engineering at the Christian-Albrechts University (CAU) of Kiel, Germany, with focus on digital communications. During his studies, he spent six months in 2000 with the Global Wireless Systems Research Group, Lucent Technologies, Bell Labs U.K., in Swindon, England. He received the Dipl.-Ing. degree from the CAU Kiel in July 2001. For his diploma thesis on space-time codes he received the Prof.-Dr.-Werner-Petersen Award. From August 2001 to October 2006 he was working toward his Ph.D. degree as a research and teaching assistant at the Information and Coding Theory Lab, CAU Kiel, and received his Ph.D. degree in December 2006. He received an award from the Friends of the Faculty of Engineering for the best dissertation in 2006. From January 2007 to December 2008 he was with the Communication Theory Group, University of British Columbia, in Vancouver, Canada, as a post-doctoral research fellow sponsored by the German Academic Exchange Service (DAAD). In March 2009, he joined EADS Cassidian Electronics, in Ulm, Germany. His research interests concern physical layer aspects of future wireless communication systems, especially multiple-antenna techniques, ultra-wideband and cognitive radio systems, relaying and cooperative diversity techniques. Dr. Mietzner has served as a TPC member for the IEEE WCNC 2009 & 2010, the IEEE Globecom 2009 & 2010, and the IEEE ICC 2011. He has also received the 2010 Best Paper Award from the German Information Technology Society (VDE/ITG).

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