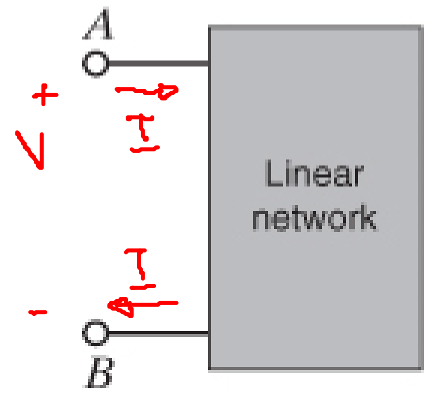
Two-Port Networks
(I&N Chap 16.1-6)

- Introduction
- Impedance/Admittance Parameters
- Hybrid Parameters
- Transmission Parameters
- Interconnecting Two-Port Networks

Single-Port Circuit

A “port” refers to a pair of terminals through which a single current flows and across which there is a single voltage.

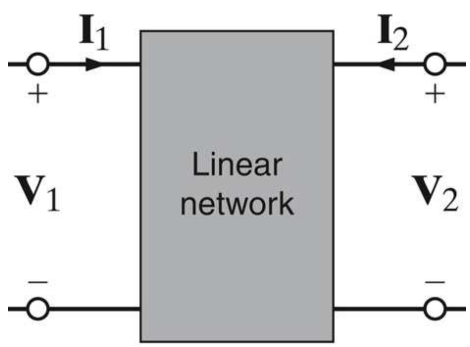


Externally, either voltage or current could be independently specified while the other quantity would be computed. The Thevenin equivalent permits a simple model of the linear network, regardless of the number of components in the network.

Two-Port

A two-port network requires two terminal pairs (total 4 terminals). Amongst the two voltages and two currents shown, generally two can be independently specified (externally). How many distinct pairs of independent quantities are possible? $C_4^2 = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{2 \times 2} = 6$

input

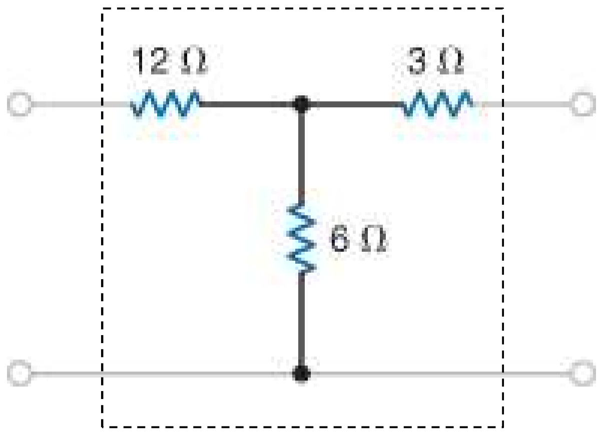


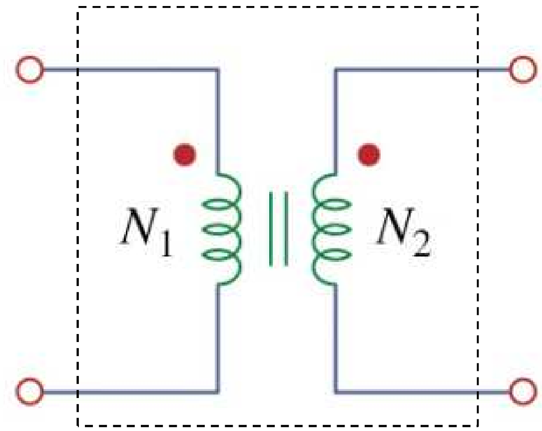
Output

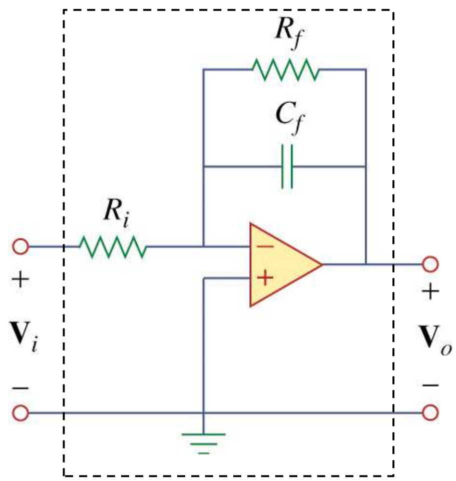
By convention, regard Port 1 as the input and Port 2 as the input (and use the polarity labels shown). We consider circuits with no internal independent sources.

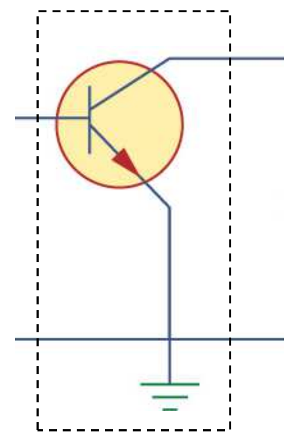
Motivating Examples

We’ve seen two-ports before.

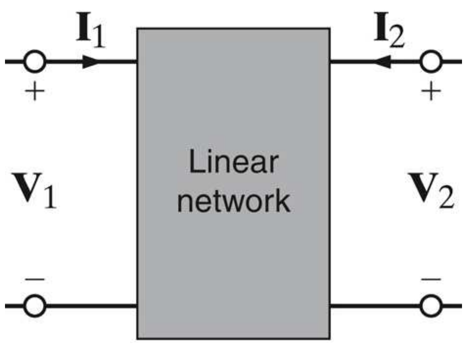








Admittance Parameters



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

→ only external independent sources!

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

How to determine y parameters:

$y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0}$ → short circuit input admitance

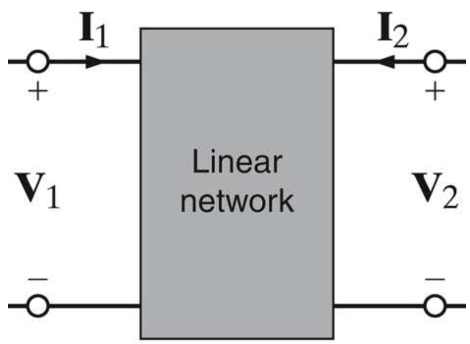
$y_{22} = \left.\dfrac{I_2}{V_2}\right|_{V_1=0}$ → short circuit output admitan

$$\left.\begin{aligned} y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \end{aligned}\right\} \rightarrow$$

transadmitances

(transfer admitances)

Impedance Parameters



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

→ independent sources

$V_1 = Z_{11} I_1 + Z_{12} I_2$

$V_2 = Z_{21} I_1 + Z_{22} I_2$

Determine Z parameters:

$Z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}$ → open circuit input impedance

$Z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$ → O.C. output impedance

$Z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$

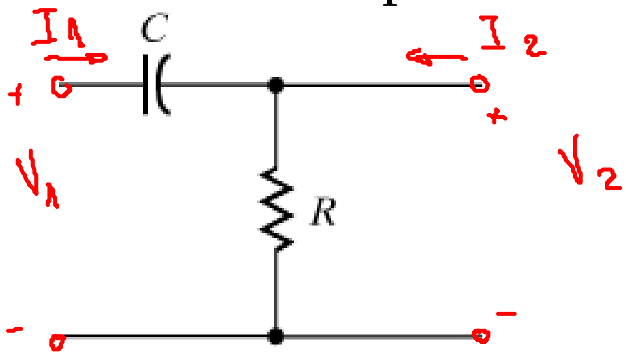
$Z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}$

→ transimpedances (transfer impedances)

Note: passive components → $Z_{12} = Z_{21}$, $y_{12} = y_{21}$ → circuit is "reciprocal"

Example

Find the two-port admittance and impedance parameters.



$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0} = sC \ ; \ y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0} = \frac{1}{R} + sC$$

$$y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0} = -sC \ ; \ y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0} = -sC$$

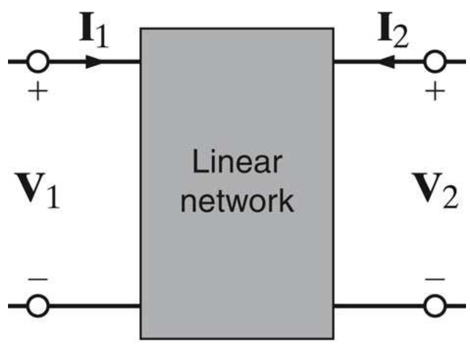
$$[Y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} sC & -sC \\ -sC & \frac{1}{R}+sC \end{bmatrix}$$

$$V_1 = z_{11}I_1 + z_{12}I_2 \Rightarrow z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0} = R + \frac{1}{sC} \ ; \ z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0} = R = z_{21}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 \Rightarrow z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0} = R$$

$$[Z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} R+\frac{1}{sC} & R \\ R & R \end{bmatrix} ; \ [Z]^{-1} = [Y] = \begin{bmatrix} R & -R \\ -R & R+\frac{1}{sC} \end{bmatrix} \times \frac{1}{R^2 + R/sC - R^2}$$

Hybrid Parameters



$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$V_1 = h_{11} I_1 + h_{12} V_2$

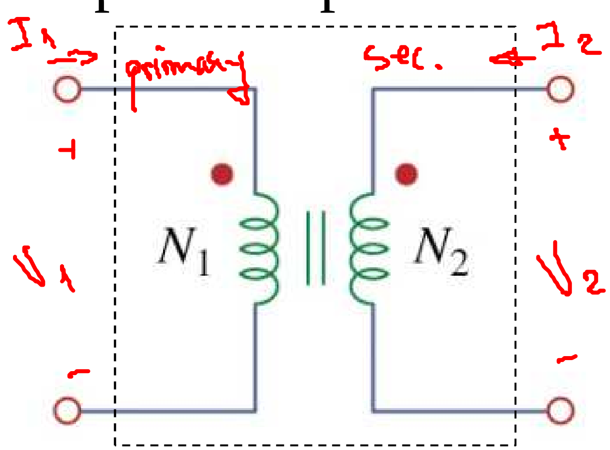
$I_2 = h_{21} I_1 + h_{22} V_2$

$h_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$ = s.c. (short circuit) input impedance

$h_{22} = \frac{I_2}{V_2}\Big|_{I_1=0}$ = o.c. output admittance

$h_{21} = \frac{I_2}{I_1}\Big|_{V_2=0}$ (s.c. forward current gain); $h_{21} = \frac{V_1}{V_2}\Big|_{I_1=0}$ = o.c. reverse voltage gain

E.g. Consider the ideal Xformer. Do the admittance or impedance parameters exist?

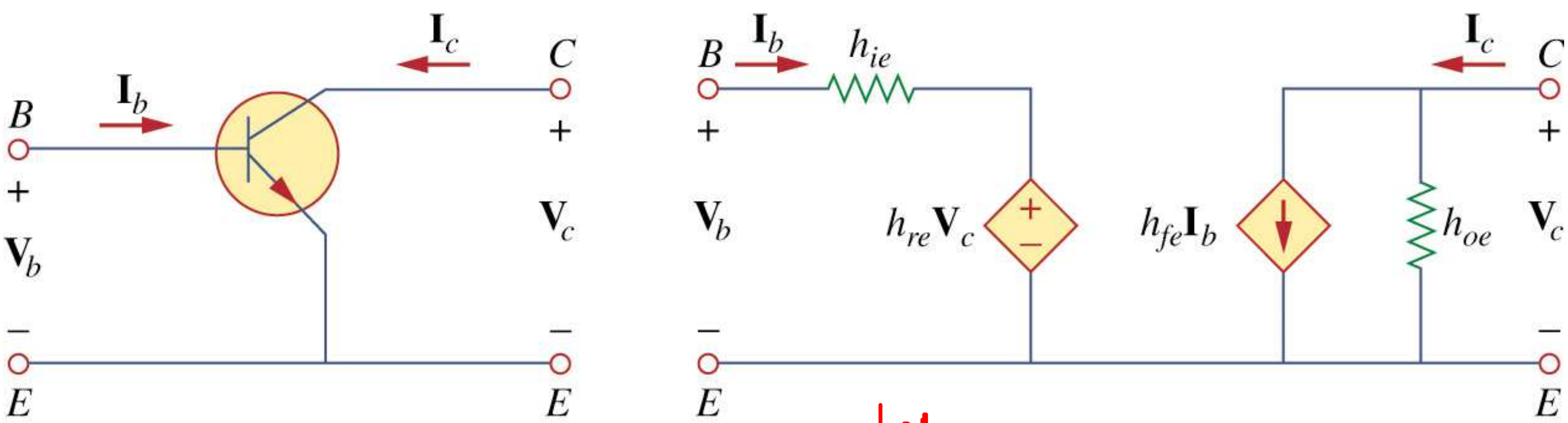


$n = \frac{N_2}{N_1} = \frac{V_2}{V_1} = -\frac{I_1}{I_2}$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{n} \\ -\frac{1}{n} & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Transistor Hybrid Parameters

The "small signal" analysis (DC analysis of quiescent point is separate) of BJTs often utilises hybrid parameters.



common emmiter

(a) (b)

$V_b = h_{ie} I_b + h_{re} V_c$

$I_c = h_{fe} I_b + h_{oe} V_c$

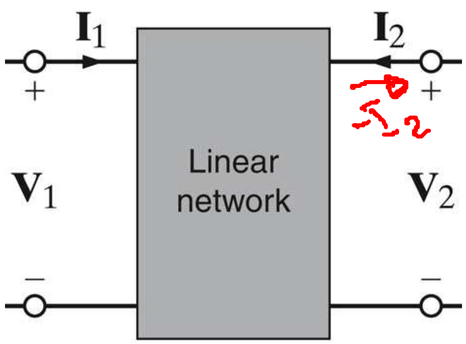
h_{ie} = base input impedance (h_{11})

$h_{re} = \frac{V_b}{V_c}\Big|_{I_b=0}$ = reverse voltage feedback ratio

h_{fe} = forward current gain (base to collector)

h_{oe} =

Transmission Parameters

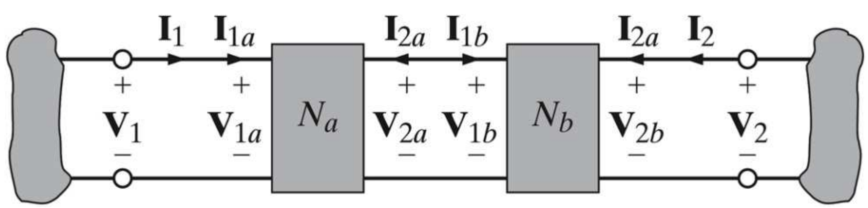


$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$V_1 = A V_2 - B I_2$

$I_1 = C V_2 - D I_2$

Transmission parameters are useful for cascaded networks.



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$= [T_a][T_b] \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$[T] = [T_a][T_b]$ – order is important!

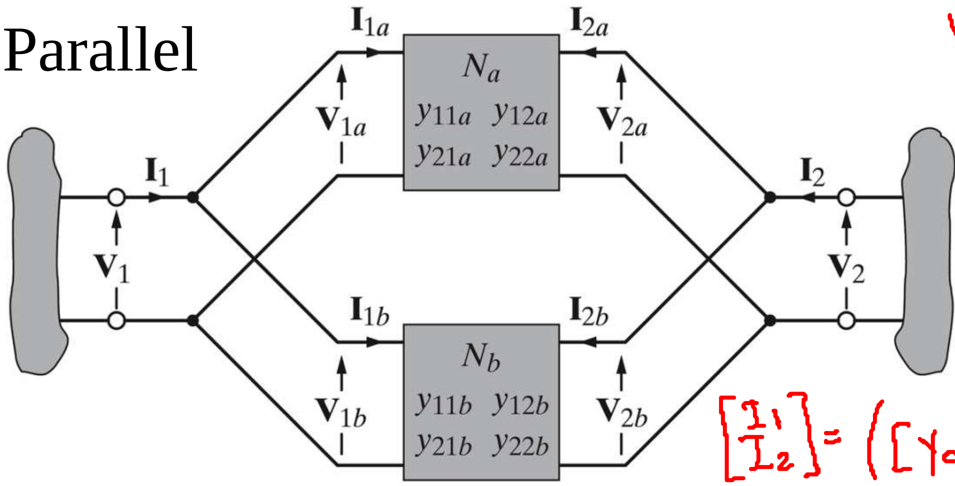
Parameter Conversions

TABLE 16.1 Two-port parameter conversion formulas

[Z]	$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta_Y} & \frac{-y_{12}}{\Delta_Y} \\ \frac{-y_{21}}{\Delta_Y} & \frac{y_{11}}{\Delta_Y} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta_H}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$
[Y]	$\begin{bmatrix} \frac{z_{22}}{\Delta_Z} & \frac{-z_{12}}{\Delta_Z} \\ \frac{-z_{21}}{\Delta_Z} & \frac{z_{11}}{\Delta_Z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{-\Delta_T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_H}{h_{11}} \end{bmatrix}$
[T]	$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_Z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_Y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta_H}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$
[h]	$\begin{bmatrix} \frac{\Delta_Z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_Y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$

Other 2-Port Interconnections

Parallel



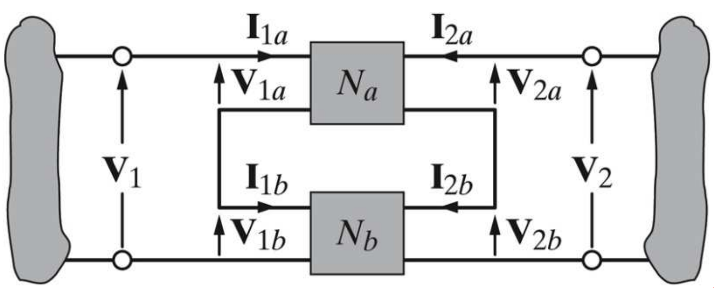
$V_1 = V_{1a} = V_{1b}$; $V_2 = V_{2a} = V_{2b}$

$I_1 = I_{1a} + I_{1b}$; $I_2 = I_{2a} + I_{2b}$

$$\begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} = [Y_a] \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} ; \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix} = [Y_b] \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \left([Y_a] + [Y_b] \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Series



$I_1 = I_{1a} = I_{1b}$; $I_2 = I_{2a} = I_{2b}$

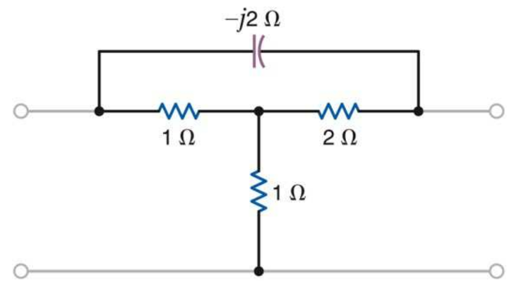
$V_1 = V_{1a} + V_{1b}$; $V_2 = V_{2a} + V_{2b}$

$$\begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} = [Z_a] \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} ; \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} = [Z_b] \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix}$$

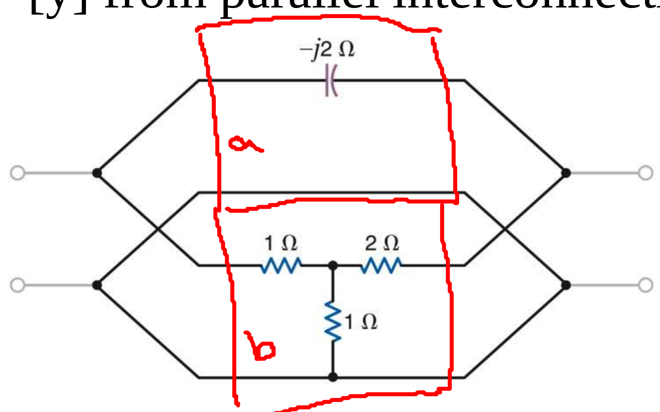
$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \left([Z_a] + [Z_b] \right) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

I&N Examples 16.5 & 16.6

For the circuit shown, use the interconnected parallel and series networks below to find, respectively, the circuit admittance and impedance parameters.



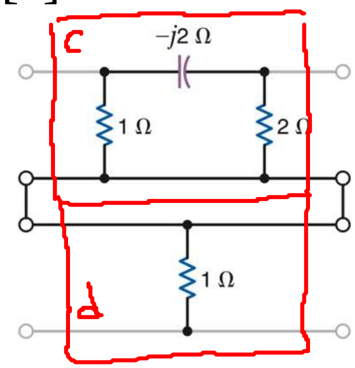
[y] from parallel interconnection



$$[y_a] = \begin{bmatrix} \frac{j}{2} & -\frac{j}{2} \\ -\frac{j}{2} & \frac{j}{2} \end{bmatrix} ; \quad [y_b] = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$[y] = [y_a] + [y_b] =$$

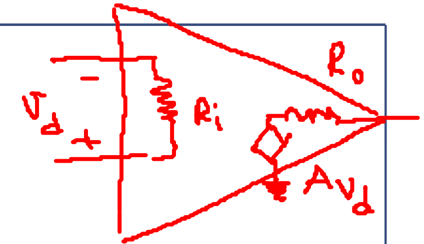
[z] from series interconnection

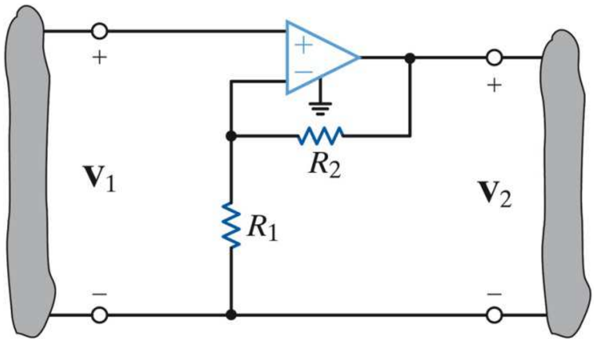


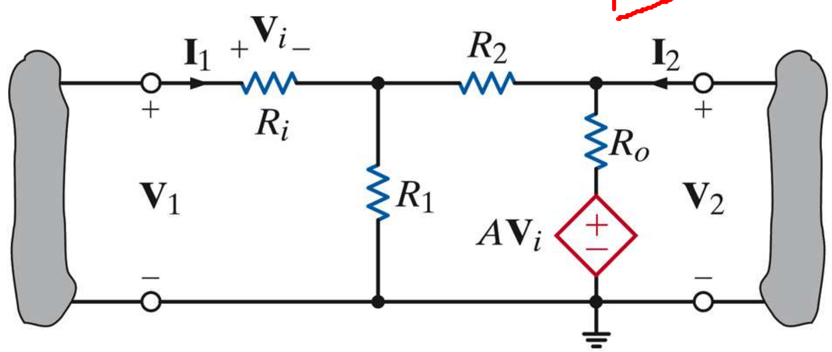
$$[z_c] = \begin{bmatrix} (2-j2) \parallel 1 & z_{12} \\ z_{21} & (1-j2) \parallel 2 \end{bmatrix}$$

$$[z_d] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} ; \quad [z] = [z_c + z_d]$$

I&N Example 16.3







Find the hybrid parameters for the circuit using the non-ideal op-amp model.

ideal op-amp : $\frac{V_2}{V_1} = 1 + \frac{R_2}{R_1}$

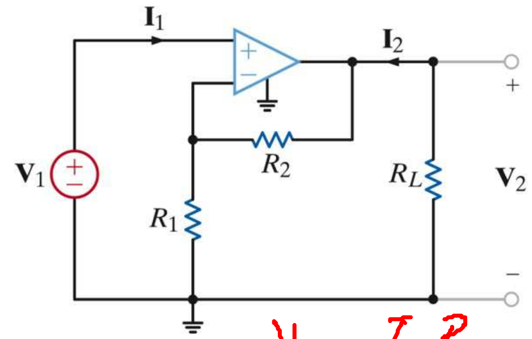
$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$; $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} I_1 \Big|_{V_2=0} = \begin{bmatrix} R_i + R_1 \| R_2 \\ -\left(\frac{R_1}{R_1+R_2} + A\frac{R_i}{R_o}\right) \end{bmatrix} I_1 \Big|_{V_2=0}$

$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} V_2 \Big|_{I_1=0} = \begin{bmatrix} \frac{R_1}{R_1+R_2} \\ \frac{1}{R_o} + \frac{1}{R_1+R_2} \end{bmatrix} V_2 \Big|_{I_1=0}$

$\Rightarrow [h] = \begin{bmatrix} \quad \end{bmatrix}$

I&N Example 16.9

Analyse the effect of load R_L on the gain and gain error using a non-ideal op-amp model (use A=20000, R_i=1 MΩ, R_o=500 Ω, R_1=1 kΩ and R_2=49 kΩ).



ideal op-amp $\Rightarrow \dfrac{V_2}{V_1} = 50 = A_{ideal}$

$V_2 = -I_2 R_L$

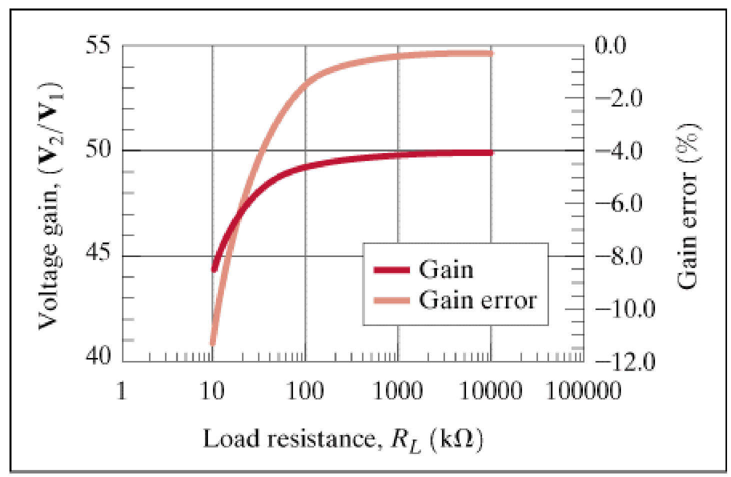
$V_1 = h_{11} I_1 + h_{12} V_2 \quad / \times h_{21}$

$-\left(I_2 = h_{21} I_1 + h_{22} V_2\right) \quad / \times h_{11}$

$h_{21} V_1 + \dfrac{h_{11} V_2}{R_L} = h_{21} h_{12} V_2 - h_{11} h_{22} V_2 = \Delta h V_2$

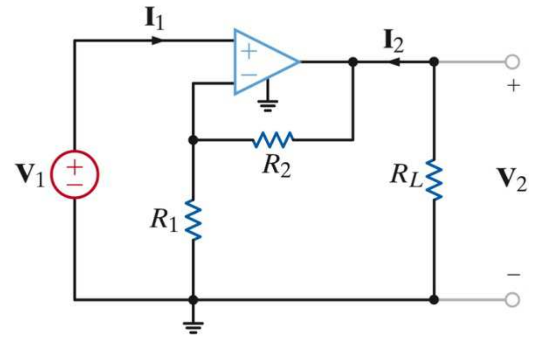
$\Rightarrow \dfrac{V_2}{V_1} = -\dfrac{h_{21}}{\Delta h + \dfrac{h_{11}}{R_L}} = \dfrac{49.9}{1 + \dfrac{1.25}{R_L}} = A_{actual}$

Gain error $= \dfrac{A_{actual} - A_{ideal}}{A_{ideal}}$



I&N Example 16.10

Using the op-amp circuit from example 16.9 but using a different value for R_2, compare the single-stage vs two-stage amplifier to achieve a gain of 10,000.



single stage : $A_V = 1 + \frac{R_2}{R_1} \Rightarrow \frac{R_2}{R_1} = 9,999$

e.g. $R_1 = 1k\Omega \Rightarrow R_2 = 9.999 M\Omega$

$$\Rightarrow [h] = \begin{bmatrix} 1.001 M\Omega & 1\times10^{-4} \\ -4\times10^{7} & 2.000\ mS \end{bmatrix} \Rightarrow \frac{V_2}{V_1} = \frac{6667}{1+\frac{166.7}{R_L}}$$

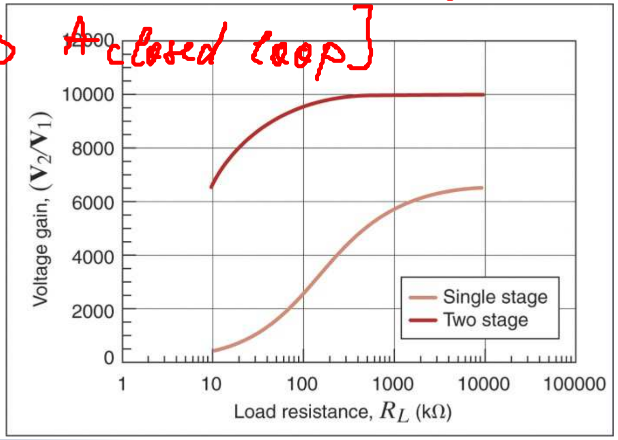
Problems : 1) $R_2 > R_i$ (ideally want $R_i \to \infty$)

2) $A_{actual} \cong 6.667$ ($R_L \to \infty$) - same order of magnitude as A_{ideal} {want $A_{open\ loop} \gg A_{closed\ loop}$}

$\Rightarrow$ two stage @ $A = 100$

$R_1 = 1k\Omega$; $R_2 = 99 k\Omega$

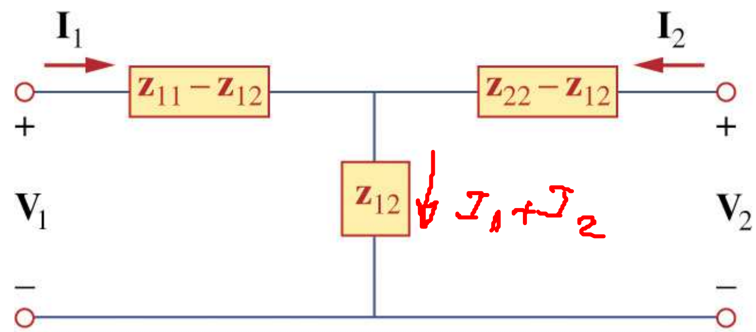
$$[h] = \begin{bmatrix} 1.001 M\Omega & 0.01 \\ -4\times10^{7} & 2\ mS \end{bmatrix} \Rightarrow \frac{V_2}{V_1} = \frac{99.5}{1+\frac{2.49}{R_L}}$$



Equivalent Circuits from [z]

A reciprocal network with known impedance parameters can be represented by the "T-network" of impedances shown.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$



$$V_1 = (Z_{11} - Z_{12}) I_1 + Z_{12} (I_1 + I_2)$$

$$= Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{12} I_1 + Z_{22} I_2$$

Summary

- A two-port network has an input port and an output port, each with each port involving a single current and a single voltage.
- If the two-port network is linear and does not contain any independent sources, it may be possible to characterize up to 6 different sets of matrix relationships. We discussed four: admittance [y], impedance [z], hybrid [h], and transmission [T]. If the parameters exist, they can be calculated or measured individually by short-circuiting or open-circuiting the appropriate port.
- A two-port network is reciprocal if $y_{12}=y_{21}$, $z_{12}=z_{21}$, $h_{12}=-h_{21}$. If the linear network only contains passive elements, it is reciprocal.
- When two-port networks are connected (a) in series, their impedance parameters add; (b) in parallel, their admittance parameters add; and (c) in cascade, their transmission parameters multiply.